ON THE OPTIMAL STOPPING PROBLEM FOR ONE–DIMENSIONAL DIFFUSIONS

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Abstract. A new characterization of excessive functions for arbitrary one–dimensional regular diffusion processes is provided, using the notion of concavity. It is shown that excessive functions are essentially concave functions in some suitable generalized sense, and vice–versa. This, in turn, permits a characterization of the value function of the optimal stopping problem as "the smallest nonnegative concave majorant of the reward function", and allows us to generalize results of Dynkin–Yushkevich for the standard Brownian motion. Moreover, we show how to reduce the discounted optimal stopping problems for an arbitrary diffusion process, to an *undiscounted* optimal stopping problem for the standard Brownian motion.

The concavity of the value functions also leads to conclusions about their smoothness, thanks to the properties of concave functions. One is thus led to a new perspective and new facts about the smooth–fit principle in the context of optimal stopping. The results are illustrated in detail on a number of non–trivial, concrete optimal stopping problems, both old and new.

1. Introduction and Summary

This paper studies the optimal stopping problem for one–dimensional diffusion processes. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a standard Brownian motion $B = \{B_t; t \geq 0\}$, and consider the diffusion process X with state space $\mathcal{I} \subseteq \mathbb{R}$ and dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \tag{1.1}$$

for some Borel functions $\mu : \mathcal{I} \to \mathbb{R}$ and $\sigma : \mathcal{I} \to (0, \infty)$. We assume that $\mathcal{I}$ is an interval with endpoints $-\infty \leq a < b \leq +\infty$, and that X is regular in (a, b); i.e., X

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reaches y with positive probability starting at x, for every x and y in (a,b). We shall denote by $\mathbb{F} = \{\mathcal{F}_t\}$ the natural filtration of X.

Let $\beta \geq 0$ be a constant, and $h(\cdot)$ be a Borel function such that $\mathbb{E}_x\big[e^{-\beta\tau}h(X_\tau)\big]$ is well–defined for every $\mathbb{F}$–stopping time τ and $x \in \mathcal{I}$. By convention

$$f(X_\tau(\omega)) = 0 \quad \text{on} \quad \{\tau = +\infty\}, \quad \text{for every Borel function } f(\cdot).$$

Finally, we denote by

$$V(x) \triangleq \sup_{\tau\in\mathcal{S}} \quad \mathbb{E}_x\big[e^{-\beta\tau}h(X_\tau)\big], \qquad x \in \mathcal{I}, \tag{1.2}$$

the *value function* of the optimal stopping problem with *reward function* $h(\cdot)$ and *discount rate* β, where the supremum is taken over the class $\mathcal{S}$ of all $\mathbb{F}$–stopping times. The optimal stopping problem is to find the value function, as well as an optimal stopping time τ^* for which the supremum is attained, if such a time exists.

One of the best–known characterizations of the value function $V(\cdot)$ is given in terms of β–excessive functions (for the process X), namely, the nonnegative functions $f(\cdot)$ that satisfy

$$f(x) \geq \mathbb{E}_x\big[e^{-\beta\tau}f(X_\tau)\big], \;\; \forall\, \tau \in \mathcal{S}, \; \forall\, x \in \mathcal{I}. \tag{1.3}$$

For every β–excessive function $f(\cdot)$ *majorizing* $h(\cdot)$, (1.3) implies that $f(x) \geq V(x)$, $x \in \mathcal{I}$. On the other hand, thanks to the strong Markov property of diffusion processes, it is not hard to show that $V(\cdot)$ is itself a β–excessive function.

Theorem 1.1 (Dynkin [4]). *The value function $V(\cdot)$ of (1.2) is the smallest β–excessive (with respect to X) majorant of $h(\cdot)$ on $\mathcal{I}$, if $h(\cdot)$ is lower semi–continuous.*

This characterization of the value function often serves as a verification tool. It does not however describe how to calculate the value function explicitly for a general diffusion process. The common practice in the literature is therefore to guess the value function, and then to put it to the test using Theorem 1.1.

One special optimal stopping problem, whose solution for arbitrary reward functions is perfectly known, was studied by Dynkin and Yushkevich [8]. These authors study the optimal stopping problem of (1.2) under the following assumptions:

$$\text{(DY)} \qquad \left\{ \begin{array}{l} X \text{ is a standard Brownian motion starting in a closed} \\ \text{bounded interval } [a,b]\text{, and is absorbed at the boundaries} \\ \text{(i.e., } \mu(\cdot) \equiv 0 \text{ on } [a,b]\text{, } \sigma(\cdot) \equiv 1 \text{ on } (a,b)\text{, and } \sigma(a) = \sigma(b) = \\ 0\text{, and } \mathcal{I} \equiv [a,b] \text{ for some } -\infty < a < b < \infty\text{). Moreover,} \\ \beta = 0\text{, and } h(\cdot) \text{ is a bounded Borel function on } [a,b]. \end{array} \right\}$$

Their solution relies on the following key theorem, which characterizes the excessive functions for one–dimensional Brownian motion.

Theorem 1.2 (Dynkin and Yushkevich [8])**.** *Every 0–excessive (or simply, excessive) function for one–dimensional Brownian motion X is concave, and vice–versa.*

In conjunction with Theorem 1.1, this result implies the following

Corollary 1.1. *The value function $V(\cdot)$ of (1.2) is the smallest nonnegative concave majorant of $h(\cdot)$ under the assumptions (DY).*

This paper generalizes the results of Dynkin and Yushkevich for the standard Brownian motion, to arbitrary one–dimensional regular diffusion processes. We show that the excessive functions for such a diffusion process X coincide with the concave functions, in some suitably generalized sense (cf. Proposition 3.1). A similar concavity result will also be established for β–excessive functions (cf. Proposition 4.1 and Proposition 5.1). These explicit characterizations of excessive functions allow us to describe the value function $V(\cdot)$ of (1.2) in terms of generalized concave functions, in a manner very similar to Theorem 1.2 (cf. Proposition 3.2 and Proposition 4.2). The new characterization of the value function, in turn, has important consequences.

The straightforward connection between generalized and ordinary concave functions, reduces the optimal stopping problem for arbitrary diffusion processes to that for the standard Brownian motion (cf. Proposition 3.3). Therefore, the "special" solution of Dynkin and Yushkevich, in fact, becomes a fundamental technique, of general applicability, for solving the optimal stopping problems for regular one–dimensional diffusion processes.

The properties of concave functions, summarized in Section 2, will help establish necessary and sufficient conditions about the finiteness of value functions and about the existence and characterization of optimal stopping times, when the diffusion process is not contained in a compact interval, or when the boundaries are not absorbing (cf. Proposition 5.2 and Proposition 5.7)

We shall also show that the concavity and minimality properties of the value function determine its smoothness. This will let us understand the major features of the method of Variational Inequalities. We offer, for example, a new exposition and, we believe, a better understanding, of the smooth–fit principle, which is crucial to this method. It is again the concavity of the value function that helps to unify many of the existing results in the literature about the smoothness of $V(\cdot)$ and the smooth–fit principle.

Preview. We overview the basic facts about one–dimensional diffusion processes and concave functions in Section 2. In Section 3 and Section 4, we solve undiscounted and discounted, respectively, stopping problems for a regular diffusion process, stopped at the time of first exit from a given closed and bounded interval. In Section 5, we study the same problem when the state–space of the diffusion process is an unbounded interval, or when the boundaries are not absorbing.

The results are used in Section 6 to treat a host of optimal stopping problems with explicit solutions, and in Section 7 to discuss further consequences of the new characterization for the value functions. We address especially the smoothness of the value function, and take a new look at the associated variational inequalities.

2. One–Dimensional Regular Diffusion Processes and Concave Functions

Let X be a one-dimensional regular diffusion of the type (1.1), on an interval $\mathcal{I}$. We shall assume that (1.1) has a (weak) solution, which is unique in the sense of the probability law. This is guaranteed, if $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy

$$\int_{(x-\varepsilon,x+\varepsilon)} \frac{1+|\mu(y)|}{\sigma^2(y)}\, dy < \infty, \quad \text{for some } \varepsilon > 0, \tag{2.1}$$

at every $x \in \mathrm{int}(\mathcal{I})$ (Karatzas and Shreve [13, 329–353]), together with precise description of the behavior of the process at the boundaries of the state–space $\mathcal{I}$. If killing is allowed at time ζ, then the dynamics in (1.1) are valid for $0 \le t < \zeta$. We shall assume, however, that X can only be killed at the endpoints of $\mathcal{I}$ which do not belong to $\mathcal{I}$.

Define $\tau_r \triangleq \inf\{t \ge 0 : X_t = r\}$ for every $r \in \mathcal{I}$. A one–dimensional diffusion process X is called *regular*, if for any $x \in \mathrm{int}(\mathcal{I})$ and $y \in \mathcal{I}$, we have $\mathbb{P}_x(\tau_y < +\infty) > 0$.

Hence, the state–space $\mathcal{I}$ cannot be decomposed into smaller sets from which X could not exit. Under the condition (2.1), the diffusion X of (1.1) is regular.

The major consequences of this assumption are listed below: their proofs can be found in Revuz and Yor [15, pages 300–312]. Let $J \triangleq (l,r)$ be a subinterval of $\mathcal{I}$ such that $[l,r] \subseteq \mathcal{I}$, and σ_J the exit time of X from J. If $x \in J$, then $\sigma_J = \tau_l \wedge \tau_r$, $\mathbb{P}_x$–a.s. For $x \notin J$, then $\sigma_J = 0$, $\mathbb{P}_x$–a.s.

Proposition 2.1. *If J is bounded, then the function $m_J(x) \triangleq \mathbb{E}_x[\sigma_J]$, $x \in I$ is bounded on J. In particular, σ_J is a.s. finite.*

Proposition 2.2. *There exists a continuous, strictly increasing function $S(\cdot)$ on $\mathcal{I}$ such that for any l, r, x in $\mathcal{I}$, with $a \le l < x < r \le b$, we have*

$$\mathbb{P}_x(\tau_r < \tau_l) = \frac{S(x)-S(l)}{S(r)-S(l)}, \qquad \text{and} \qquad \mathbb{P}_x(\tau_l < \tau_r) = \frac{S(r)-S(x)}{S(r)-S(l)}. \tag{2.2}$$

Any other function $\widetilde{S}$ with the same properties is an affine transformation of S, i.e., $\widetilde{S} = \alpha S + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. The function S is unique in this sense, and is called the "scale function" of X.

If the killing time ζ is finite with positive probability, and $\lim_{t\uparrow\zeta} X_t = a$ (say), then $\lim_{x\to a} S(x)$ is finite. We shall define $S(a) \triangleq \lim_{x\to a} S(x)$, and set $S(X_\zeta) = S(l)$. With this in mind, we have:

Proposition 2.3. *A locally bounded Borel function f is a scale function, if and only if the process $Y_t^f \triangleq f(X_{t\wedge\zeta\wedge\tau_a\wedge\tau_b})$, $t \ge 0$, is a local martingale. Furthermore, if X can be represented by the stochastic differential equation (1.1), then*

$$S(x) = \int_c^x \exp\left\{-\int_c^y \frac{2\mu(z)}{\sigma^2(z)}dz\right\} dy, \qquad x \in \mathcal{I}, \tag{2.3}$$

for any arbitrary but fixed $c \in \mathcal{I}$.

The scale function $S(\cdot)$ has derivative $S'(x) = \exp\left\{\int_c^x [-2\mu(u)/\sigma^2(u)]\, du\right\}$ on $\operatorname{int}(\mathcal{I})$, and we shall define

$$S''(x) \triangleq -\frac{2\mu(x)}{\sigma^2(x)}\, S'(x), \quad x \in \operatorname{int}(\mathcal{I}).$$

This way $\mathcal{A}S(\cdot) \equiv 0$, where the second–order differential operator

$$\mathcal{A}u(\cdot) \triangleq \frac{1}{2}\sigma^2(\cdot)\frac{d^2u}{dx^2}(\cdot) + \mu(\cdot)\frac{du}{dx}(\cdot), \quad \text{on } \mathcal{I}, \tag{2.4}$$

is the infinitesimal generator of X. The ordinary differential equation $\mathcal{A}u = \beta u$ has two linearly independent, positive solutions. These are uniquely determined up to multiplication, if we require one of them to be strictly increasing and the other to be strictly decreasing. We shall denote the *increasing* solution by $\psi(\cdot)$ and the *decreasing* solution by $\varphi(\cdot)$. In fact, we have

$$\psi(x) = \begin{cases} \mathbb{E}_x[e^{-\beta\tau_c}], & \text{if } x \le c \\ \dfrac{1}{\mathbb{E}_c[e^{-\beta\tau_x}]}, & \text{if } x > c \end{cases}, \quad \varphi(x) = \begin{cases} \dfrac{1}{\mathbb{E}_c[e^{-\beta\tau_x}]}, & \text{if } x \le c \\ \mathbb{E}_x[e^{-\beta\tau_c}], & \text{if } x > c \end{cases}, \tag{2.5}$$

for every $x \in \mathcal{I}$, and arbitrary but fixed $c \in \mathcal{I}$ (cf. Itô and McKean [10, pages 128–129]). Solutions of $\mathcal{A}u = \beta u$ in the domain of infinitesimal operator $\mathcal{A}$ are obtained as linear combinations of $\psi(\cdot)$ and $\varphi(\cdot)$, subject to appropriate boundary conditions imposed on the process X. If an endpoint is contained in the state–space $\mathcal{I}$, we shall assume that it is absorbing; and if it is not contained in $\mathcal{I}$, we shall assume that X is killed if it can reach the boundary with positive probability. In either case, the boundary conditions on $\psi(\cdot)$ and $\varphi(\cdot)$ are $\psi(a) = \varphi(b) = 0$. For the complete characterization of $\psi(\cdot)$ and $\varphi(\cdot)$ corresponding to other types of boundary behavior, refer to Itô and McKean [10, pages 128–135]. Note that the *Wronskian determinant*

$$W(\psi,\varphi) \triangleq \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x) \tag{2.6}$$

of $\psi(\cdot)$ and $\varphi(\cdot)$ is a positive constant. One last useful expression is

$$\mathbb{E}_x\left[e^{-\beta\tau_y}\right] = \begin{cases} \dfrac{\psi(x)}{\psi(y)}, & x \le y \\ \dfrac{\varphi(x)}{\varphi(y)}, & x > y \end{cases}. \tag{2.7}$$

Concave Functions. Let $F : [c,d] \to \mathbb{R}$ be a strictly increasing function. A real–valued function u is called F–concave on $[c,d]$ if, for every $a \le l < r \le b$ and $x \in [l,r]$, we have

$$u(x) \ge u(l)\frac{F(r)-F(x)}{F(r)-F(l)} + u(r)\frac{F(x)-F(l)}{F(r)-F(l)}. \tag{2.8}$$

Here are some facts about the properties of F–concave functions (Dynkin [5, pages 231–240], Karatzas and Shreve [13, pages 213–214], Revuz and Yor [15, pages 544–547]).

Proposition 2.4. *Suppose $u(\cdot)$ is real-valued and F–concave, and $F(\cdot)$ is continuous on $[c,d]$. Then $u(\cdot)$ is continuous in (c,d) and $u(c) \leq \liminf_{x\downarrow c} u(x)$, $u(d) \leq \liminf_{x\uparrow d} u(x)$.*

Proposition 2.5. *Let $(u_\alpha)_{\alpha\in\Lambda}$ is a family of F–concave functions on $[c,d]$. Then $u \triangleq \wedge_{\alpha\in\Lambda} u_\alpha$ is also F–concave on $[c,d]$.*

Let $v : [c,d] \to \mathbb{R}$ be any function. Define

$$D_F^+ v(x) \equiv \frac{d^+v}{dF}(x) \triangleq \lim_{y\downarrow x} \frac{v(x)-v(y)}{F(x)-F(y)}, \quad \text{and} \quad D_F^- v(x) \equiv \frac{d^-v}{dF}(x) \triangleq \lim_{y\uparrow x} \frac{v(x)-v(y)}{F(x)-F(y)},$$

provided that limits exist. If $D_F^\pm v(x)$ exist and are equal, then $v(\cdot)$ is said to be F-differentiable at x, and we write $D_F v(x) = D_F^\pm v(x)$.

Proposition 2.6. *Suppose $u : [c,d] \to \mathbb{R}$ is F–concave. Then we have the following:*

(i) *The derivatives $D_F^+ u(\cdot)$ and $D_F^- u(\cdot)$ exist in (c,d). Both are non–increasing and $D_F^+ u(l) \geq D_F^- u(x) \geq D_F^+ u(x) \geq D_F^- u(r)$, for every $c < l < x < r < d$.*

(ii) *Let $x_0 \in (c,d)$. For every $D_F^+ u(x_0) \leq \theta \leq D_F^- u(x_0)$, we have $u(x_0) + \theta[F(x) - F(x_0)] \geq u(x)$, $\forall x \in [c,d]$.*

(iii) *If $F(\cdot)$ is continuous on $[c,d]$, then $D_F^+ u(\cdot)$ is right–continuous, and $D_F^- u(\cdot)$ is left–continuous. The derivatives $D_F^\pm u(\cdot)$ have the same set of continuity points; in particular, except for x in a countable set N, we have $D_F^+ u(x) = D_F^- u(x)$.*

3. Undiscounted Optimal Stopping

Suppose we start the diffusion process X of (1.1) in a closed and bounded interval $[c,d]$ contained in the interior of the state–space $\mathcal{I}$, and stop X as soon as it reaches one of the boundaries c or d. For a given Borel–measurable and bounded function $h : [c,d] \to \mathbb{R}$, we set

$$V(x) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}_x[h(X_\tau)], \quad x \in [c,d]. \tag{3.1}$$

The question is to characterize the function $V(\cdot)$, and to find an optimal stopping time τ^* such that $V(x) = \mathbb{E}_x[h(X_{\tau^*})]$, $x \in [c,d]$, if such τ^* exists. If $h(\cdot) \leq 0$, then trivially $V \equiv 0$, and $\tau \equiv \infty$ is an optimal stopping time. Therefore, we shall assume $\sup_{x\in[c,d]} h(x) > 0$.

Following Dynkin and Yushkevich [8, pages 112–126], we shall first characterize the class of excessive functions. These play a fundamental role in optimal stopping problems, as shown in Theorem 1.1

To motivate what follows, let $U : [c, d] \to \mathbb{R}$ be an excessive function of X. For any stopping time τ of X, and $x \in [c, d]$, we have $U(x) \geq \mathbb{E}_x[U(X_\tau)]$. In particular, if $x \in [l, r] \subseteq [c, d]$, we may take $\tau = \tau_l \wedge \tau_r$, where $\tau_r \triangleq \inf\{t \geq 0 : X_t = r\}$, and then the regularity of X gives

$$U(x) \geq \mathbb{E}_x[U(X_{\tau_l \wedge \tau_r})] = U(l) \cdot \mathbb{P}_x(\tau_l < \tau_r) + U(r) \cdot \mathbb{P}_x(\tau_l > \tau_r), \qquad x \in [l, r].$$

With the help of (2.2), the above inequality becomes

$$U(x) \geq U(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + U(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)}, \qquad x \in [l, r]. \tag{3.2}$$

In other words, every excessive function of X is S–concave on $[c, d]$ (see Section 2 for a discussion). When X is a standard Brownian motion, Dynkin and Yushkevich [8] showed that the reverse is also true; we shall show next that the reverse is true for an *arbitrary* diffusion process X.

Let $S(\cdot)$ be the scale function of X as above, and recall that $S(\cdot)$ is real–valued, strictly increasing and continuous on $\mathcal{I}$.

Proposition 3.1 (Characterization of Excessive Functions)**.** *A function $U : [c, d] \to \mathbb{R}$ is nonnegative and S–concave on $[c, d]$, if and only if*

$$U(x) \geq \mathbb{E}_x[U(X_\tau)], \qquad \forall\, \tau \in \mathcal{S}, \forall\, x \in [c, d]. \tag{3.3}$$

This, in turn, allows us to conclude the main result of this section, namely

Proposition 3.2 (Characterization of the Value Function)**.** *The value function $V(\cdot)$ of (3.1) is the smallest nonnegative, S–concave majorant of $h(\cdot)$ on $[c, d]$.*

We defer the proofs of Proposition 3.1 and Proposition 3.2 to the end of the section, and discuss their implications first.

It is usually a simple matter to find the smallest nonnegative concave majorant of a bounded function on some closed bounded interval: It coincides geometrically with the rope stretched from above the graph of function, with both ends pulled to the ground. On the contrary, it is hard to visualize the nonnegative S–concave majorant of a function. The following Proposition has therefore some importance, when we need to calculate $V(\cdot)$ explicitly; it was already noticed by Karatzas and Sudderth [14].

Proposition 3.3. *On the interval $[S(c), S(d)]$, let $W(\cdot)$ be the smallest nonnegative concave majorant of the function $H(y) \triangleq h\big(S^{-1}(y)\big)$. Then we have $V(x) = W\big(S(x)\big)$, for every $x \in [c, d]$.*

The concave characterization of Proposition 3.2 for the value function allows us to obtain information about the smoothness of $V(\cdot)$ and the existence of an optimal stopping time. Consider *the optimal stopping region*

$$\mathbf{\Gamma} \triangleq \{x \in [c, d] : V(x) = h(x)\} \qquad \text{and define} \qquad \tau^* \triangleq \inf\{t \geq 0 : X_t \in \mathbf{\Gamma}\}, \tag{3.4}$$

the time of first–entry into this region. The proof of the following result is similar to that in Dynkin and Yushkevich [8, pages 112–119].

Proposition 3.4. *If $h(\cdot)$ is continuous on $[c, d]$, then so is $V(\cdot)$, and the stopping time τ^* of (3.4) is optimal.*

Remark 3.1. Since the standard Brownian motion B is in *natural scale*, i.e., $S(x) = x$ up to some affine transformation, $W(\cdot)$ of Proposition 3.3 is itself the value function of some optimal stopping problem of the standard Brownian motion, namely

$$W(y) = \sup_{\tau \geq 0} \mathbb{E}_y[H(B_\tau)] = \sup_{\tau \geq 0} \mathbb{E}_y\Big[h\Big(S^{-1}\big(B_\tau\big)\Big)\Big], \quad y \in [S(c), S(d)]. \tag{3.5}$$

where the supremum is taken over all stopping times of B. Therefore, solving the original optimal stopping problem is the same as solving another, with a different reward function, but for a standard Brownian motion. If, moreover, we denote the optimal stopping region of this problem by $\widetilde{\mathbf{\Gamma}} \triangleq \{y \in [S(c), S(d)] : W(y) = H(y)\}$, then $\mathbf{\Gamma} = S^{-1}\big(\widetilde{\mathbf{\Gamma}}\big)$.

PROOF OF PROPOSITION 3.1. We have already seen in (3.2) that excessivity implies S–concavity. For the converse, suppose $U : [c, d] \to [0, +\infty)$ is S–concave; then it is enough to show

$$U(x) \geq \mathbb{E}_x[U(X_t)], \quad \forall\, x \in [c, d],\ \forall\, t \geq 0. \tag{3.6}$$

Indeed, observe that, the inequality (3.6) and the Markov property of X imply that $\{U(X_t)\}_{t \in [0, +\infty)}$ is a nonnegative supermartingale, and (3.3) follows from Optional Sampling. To prove (3.6), let us first show

$$U(x) \geq \mathbb{E}_x[U(X_{\rho \wedge t})], \quad \forall\, x \in [c, d],\ \forall\, t \geq 0, \tag{3.7}$$

where the stopping time $\rho \triangleq \tau_c \wedge \tau_d$ is the first exit time of X from (c, d).

First, note that (3.7) holds as equality at the absorbing boundary points $x = c$ and $x = d$. Next, fix any $x_0 \in (c, d)$; since $U(\cdot)$ is S–concave on $[c, d]$, Proposition 2.6(ii) shows that there exists an affine transformation $L(\cdot) = c_1 S(\cdot) + c_2$ of the scale function $S(\cdot)$, such that

$$L(x_0) = U(x_0), \qquad \text{and} \qquad L(x) \geq U(x), \;\; \forall\, x \in [c, d].$$

Thus, for any $t \geq 0$, we have

$$\mathbb{E}_{x_0}[U(X_{\rho\wedge t})] \leq \mathbb{E}_{x_0}[L(X_{\rho\wedge t})] = \mathbb{E}_{x_0}[c_1 S(X_{\rho\wedge t}) + c_2] = c_1 \mathbb{E}_{x_0}[S(X_{\rho\wedge t})] + c_2.$$

But $S(\cdot)$ is continuous on the closed and bounded interval $[c, d]$, and the process $S(X_t)$ is a continuous local martingale; so the *stopped* process $\{S(X_{\rho\wedge t}),\, t \geq 0\}$ is a bounded martingale, and $\mathbb{E}_{x_0}[S(X_{\rho\wedge t})] = S(x_0)$, for every $t \geq 0$, by optional sampling. Thus

$$\mathbb{E}_{x_0}[U(X_{\rho\wedge t})] \leq c_1 \mathbb{E}_{x_0}[S(X_{\rho\wedge t})] + c_2 = c_1 S(x_0) + c_2 = L(x_0) = U(x_0),$$

and (3.7) is proved. To show (3.6), observe that since $X_t = X_\sigma$ on $\{t \geq \sigma\}$, (3.7) implies $\mathbb{E}_x[U(X_t)] = \mathbb{E}_x[U(X_{\rho\wedge t})] \leq U(x)$, for every $x \in [c, d]$ and $t \geq 0$. □

PROOF OF PROPOSITION 3.2. Since $\tau \equiv \infty$ and $\tau \equiv 0$ are stopping times, we have $V \geq 0$ and $V \geq h$, respectively. Hence $V(\cdot)$ is nonnegative and majorizes $h(\cdot)$. To show that $V(\cdot)$ is S–concave, we shall fix some $x \in [l, r] \subseteq [c, d]$. Since $h(\cdot)$ is bounded, $V(\cdot)$ is finite on $[c, d]$. Therefore, for any arbitrarily small $\varepsilon > 0$, we can find stopping times σ_l and σ_r such that

$$\mathbb{E}_y[h(X_{\sigma_y})] \geq V(y) - \varepsilon, \quad y = l, r.$$

Define a new stopping time

$$\tau \triangleq \begin{cases} \tau_l + \sigma_l \circ \theta_{\tau_l}, & \text{on } \{\tau_l < \tau_r\}, \\ \tau_r + \sigma_r \circ \theta_{\tau_r}, & \text{on } \{\tau_l > \tau_r\}, \end{cases}$$

where θ_t is the shift operator (see Karatzas and Shreve [13, page 77 and 83]). Using the strong Markov property of X, we obtain

$$\begin{aligned}
V(x) \geq \mathbb{E}_x[h(X_\tau)] &= \mathbb{E}_x\big[h(X_\tau)1_{\{\tau_l<\tau_r\}}\big] + \mathbb{E}_x\big[h(X_\tau)1_{\{\tau_l>\tau_r\}}\big] \\
&= \mathbb{E}_x\Big[1_{\{\tau_l<\tau_r\}}\mathbb{E}_{X_{\tau_l}}\big[h(X_{\sigma_l})\big]\Big] + \mathbb{E}_x\Big[1_{\{\tau_l>\tau_r\}}\mathbb{E}_{X_{\tau_r}}\big[h(X_{\sigma_r})\big]\Big] \\
&= \mathbb{E}_l\big[h(X_{\sigma_l})\big]\mathbb{P}_x\{\tau_l<\tau_r\} + \mathbb{E}_r\big[h(X_{\sigma_r})\big]\mathbb{P}_x\{\tau_l>\tau_r\} \\
&= \mathbb{E}_l\big[h(X_{\sigma_l})\big]\frac{S(r)-S(x)}{S(r)-S(l)} + \mathbb{E}_r\big[h(X_{\sigma_r})\big]\frac{S(x)-S(l)}{S(r)-S(l)} \\
&\geq V(l)\cdot\frac{S(r)-S(x)}{S(r)-S(l)} + V(r)\cdot\frac{S(x)-S(l)}{S(r)-S(l)} - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $V(\cdot)$ is indeed a nonnegative S–concave majorant of $h(\cdot)$ on $[c,d]$.

Now, let $U : [c,d] \to \mathbb{R}$ be any other nonnegative S–concave majorant of $h(\cdot)$ on $[c,d]$. Then, Proposition 3.1 implies $U(x) \geq \mathbb{E}_x[U(X_\tau)] \geq \mathbb{E}_x[h(X_\tau)]$, for every $x \in [c,d]$ and every stopping time $\tau \in \mathcal{S}$. Therefore $U \geq V$ on $[c,d]$. This completes the proof. □

PROOF OF PROPOSITION 3.3. Trivially, $\widehat{V}(x) \triangleq W\big(S(x)\big)$, $x \in [c,d]$, is a nonnegative concave majorant of $h(\cdot)$ on $[c,d]$. Therefore $\widehat{V}(x) \geq V(x)$ for every $x \in [c,d]$.

On the other hand, $\widehat{W}(y) \triangleq V\big(S^{-1}(y)\big)$ is a nonnegative S–concave majorant of $H(\cdot)$ on $[S(c),S(d)]$. Therefore $\widehat{W}(\cdot) \geq W(\cdot)$ on $[S(c),S(d)]$, and $V(x) = \widehat{W}(S(x)) \geq W\big(S(x)\big) = \widehat{V}(x)$, for every $x \in [c,d]$. □

4. Discounted Optimal Stopping

Let us try now to see, how the results of Section 3 can be extended to study of the *discounted* optimal stopping problem

$$V(x) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}_x[e^{-\beta\tau}h(X_\tau)], \qquad x \in [c,d], \tag{4.1}$$

with $\beta > 0$. The diffusion process X and the reward function $h(\cdot)$ have the same properties as described in Section 3. Namely, X is started in a bounded closed interval $[c,d]$ contained in the interior of its state space $\mathcal{I}$, and is absorbed whenever it reaches c or d. Moreover, $h : [c,d] \to \mathbb{R}$ is a bounded, Borel–measurable function with $\sup_{x\in[c,d]} h(x) > 0$.

In order to motivate the key result of Proposition 4.1, let $U : [c,d] \to \mathbb{R}$ be a β–excessive function with respect to X. Namely, for every stopping time τ of X, and $x \in [c,d]$, we have $U(x) \geq \mathbb{E}_x\big[e^{-\beta\tau}U(X_\tau)\big]$. For a stopping time of the form $\tau = \tau_l \wedge \tau_r$, the first exit time of X from an interval $[l,r] \subseteq [c,d]$, the regularity of X implies

$$\begin{aligned} U(x) &\geq \mathbb{E}_x\big[e^{-\beta(\tau_l\wedge\tau_r)}U(\tau_l \wedge \tau_r)\big] \\ &= U(l)\cdot\mathbb{E}_x\big[e^{-\beta\tau_l}1_{\{\tau_l<\tau_r\}}\big] + U(r)\cdot\mathbb{E}_x\big[e^{-\beta\tau_r}1_{\{\tau_l>\tau_r\}}\big], \quad x \in [l,r]. \end{aligned} \tag{4.2}$$

The function $u_1(x) \triangleq \mathbb{E}_x\big[e^{-\beta\tau_l}1_{\{\tau_l<\tau_r\}}\big]$ (respectively, $u_2(x) \triangleq \mathbb{E}_x\big[e^{-\beta\tau_r}1_{\{\tau_l>\tau_r\}}\big]$) is the unique solution of $\mathcal{A}u = \beta u$ in (l,r), with boundary conditions $u_1(l) = 1$, $u_1(r) = 0$ (respectively, with $u_2(l) = 0$, $u_2(r) = 1$). In terms of the functions $\psi(\cdot)$, $\varphi(\cdot)$ of (2.5), using the appropriate boundary conditions, one calculates

$$u_1(x) = \frac{\psi(x)\varphi(r) - \psi(r)\varphi(x)}{\psi(l)\varphi(r) - \psi(r)\varphi(l)}, \quad u_2(x) = \frac{\psi(l)\varphi(x) - \psi(x)\varphi(l)}{\psi(l)\varphi(r) - \psi(r)\varphi(l)}, \quad x \in [l,r]. \tag{4.3}$$

Substituting these into the inequality (4.2) above, then dividing both sides of the inequality by $\varphi(x)$ (respectively, by $\psi(x)$), we obtain

$$\frac{U(x)}{\varphi(x)} \geq \frac{U(l)}{\varphi(l)}\cdot\frac{F(r)-F(x)}{F(r)-F(l)} + \frac{U(r)}{\varphi(r)}\cdot\frac{F(x)-F(l)}{F(r)-F(l)} \qquad x \in [l,r], \tag{4.4}$$

and

$$\frac{U(x)}{\psi(x)} \geq \frac{U(l)}{\varphi(l)}\cdot\frac{G(r)-G(x)}{G(r)-G(l)} + \frac{U(r)}{\varphi(r)}\cdot\frac{G(x)-G(l)}{G(r)-G(l)}, \qquad x \in [l,r], \tag{4.5}$$

respectively, where the functions

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)}, \quad \text{and} \quad G(x) \triangleq -\frac{1}{F(x)} = -\frac{\varphi(x)}{\psi(x)}, \quad x \in [c,d] \tag{4.6}$$

are both well-defined and strictly increasing. Observe now that the inequalities (4.4) and (4.5) imply that $U(\cdot)/\varphi(\cdot)$ is F–concave, and $U(\cdot)/\psi(\cdot)$ is G–concave on $[c,d]$ (cf. Section 2). In Proposition 4.1 below, we shall show that *the converse is also true.*

It is worth pointing out the correspondence between the roles of the functions $S(\cdot)$ and 1 in the *undiscounted* optimal stopping, and the roles of $\psi(\cdot)$ and $\varphi(\cdot)$ in the *discounted* optimal stopping. Both pairs $(S(\cdot),1)$ and $(\psi(\cdot),\varphi(\cdot))$ consist of an increasing and a decreasing solution of the second–order differential equation $\mathcal{A}u = \beta u$ in $\mathcal{I}$, for the undiscounted (i.e., $\beta = 0$) and the discounted (i.e., $\beta > 0$) versions of the same optimal stopping problems, respectively. Therefore, the results of Section 3 can be restated and proved with only minor (and rather obvious) changes. Here is the key result of the section:

Proposition 4.1 (Characterization of β–excessive functions)**.** *For a given function $U : [c,d] \to [0,+\infty)$, the quotient $U(\cdot)/\varphi(\cdot)$ is an F–concave (equivalently, $U(\cdot)/\psi(\cdot)$ is a G–concave) function, if and only if $U(\cdot)$ is β–excessive, i.e.,*

$$U(x) \geq \mathbb{E}_x[e^{-\beta\tau}U(X_\tau)], \qquad \forall\, \tau \in \mathcal{S},\ \forall\, x \in [c,d]. \tag{4.7}$$

Proposition 4.2 (Characterization of the value function)**.** *The value function $V(\cdot)$ of (4.1) is the smallest nonnegative majorant of $h(\cdot)$ such that $V(\cdot)/\varphi(\cdot)$ is F–concave (equivalently, $V(\cdot)/\psi(\cdot)$ is G–concave) on [c,d].*

The equivalence of the characterizations, in Proposition 4.1 and Proposition 4.2 in terms of F and G, follows now from the definition of concave functions.

Lemma 4.1. *Let $U : [c,d] \to \mathbb{R}$ any function. Then $U(\cdot)/\varphi(\cdot)$ is F–concave on $[c,d]$, if and only if $U(\cdot)/\psi(\cdot)$ is G–concave on $[c,d]$.*

Since it is hard to visualize the nonnegative F– or G–concave majorant of a function geometrically, it will again be convenient to have a description in terms of ordinary concave functions.

Proposition 4.3. *Let $W(\cdot)$ be the smallest nonnegative concave majorant of $H \triangleq (h/\varphi) \circ F^{-1}$ on $[F(c), F(d)]$, where $F^{-1}(\cdot)$ is the inverse of the strictly increasing function $F(\cdot)$ in (4.6). Then $V(x) = \varphi(x)\, W\big(F(x)\big)$, for every $x \in [c,d]$.*

Just as in Dynkin and Yushkevich [8, pages 112–126], the continuity of the functions $\varphi(\cdot)$, $F(\cdot)$, and the F–concavity of $V(\cdot)/\varphi(\cdot)$ imply the following.

Lemma 4.2. *If $h(\cdot)$ is continuous on $[c,d]$, then $V(\cdot)$ is also continuous on $[c,d]$.*

We shall next characterize the optimal stopping rule. Define the "optimal stopping region"

$$\mathbf{\Gamma} \triangleq \{x \in [c,d] : V(x) = h(x)\}, \qquad \text{and} \qquad \tau^* \triangleq \inf\{t \geq 0 : X_t \in \mathbf{\Gamma}\}. \tag{4.8}$$

Lemma 4.3. *Let $\tau_r \triangleq \inf\{t \geq 0 : X_t = r\}$. Then for every $c \leq l < x < r \leq d$,*

$$\begin{aligned}
\mathbb{E}_x[e^{-\beta(\tau_l \wedge \tau_r)}h(X_{\tau_l \wedge \tau_r})] &= \varphi(x)\left[\frac{h(l)}{\varphi(l)} \cdot \frac{F(r)-F(x)}{F(r)-F(l)} + \frac{h(r)}{\varphi(r)} \cdot \frac{F(x)-F(l)}{F(r)-F(l)}\right], \\
&= \psi(x)\left[\frac{h(l)}{\psi(l)} \cdot \frac{G(r)-G(x)}{G(r)-G(l)} + \frac{h(r)}{\psi(r)} \cdot \frac{G(x)-G(l)}{G(r)-G(l)}\right].
\end{aligned}$$

Furthermore,

$$\mathbb{E}_x[e^{-\beta\tau_r}h(X_{\tau_r})] = \varphi(x)\frac{h(r)}{\varphi(r)}\cdot\frac{F(x)-F(c)}{F(r)-F(c)} = \psi(x)\frac{h(r)}{\psi(r)}\cdot\frac{G(x)-G(c)}{G(r)-G(c)},$$

and

$$\mathbb{E}_x[e^{-\beta\tau_l}h(X_{\tau_l})] = \varphi(x)\frac{h(l)}{\varphi(l)}\cdot\frac{F(d)-F(x)}{F(d)-F(l)} = \psi(x)\frac{h(l)}{\psi(l)}\cdot\frac{G(d)-G(x)}{G(d)-G(l)}.$$

Proof. The first and second equalities are obtained after rearranging the equation

$$\mathbb{E}_x[e^{-\beta(\tau_l\wedge\tau_r)}h(X_{\tau_l\wedge\tau_r})] \quad = \quad h(l)\cdot\mathbb{E}_x[e^{-\beta\tau_l}1_{\{\tau_l<\tau_r\}}] + h(r)\cdot\mathbb{E}_x[e^{-\beta\tau_r}1_{\{\tau_l>\tau_r\}}],$$

where $u_1(x) \triangleq \mathbb{E}_x[e^{-\beta\tau_l}1_{\{\tau_l<\tau_r\}}]$ and $u_2(x) \triangleq \mathbb{E}_x[e^{-\beta\tau_r}1_{\{\tau_l>\tau_r\}}]$ are given by (4.3). The others follow similarly. □

Proposition 4.4. *If h is continuous on $[c,d]$, then τ^* of (4.8) is an optimal stopping rule.*

Proof. Define $U(x) \triangleq \mathbb{E}_x[e^{-\beta\tau^*}h(X_{\tau^*})]$, for every $x \in [c,d]$. We have obviously $V(\cdot) \geq U(\cdot)$. To show the reverse inequality, it is enough to prove that $U(\cdot)/\varphi(\cdot)$ is a nonnegative F–concave majorant of $h(\cdot)/\varphi(\cdot)$. By adapting the arguments in Dynkin and Yushkevich [8, pages 112–126], and by using Lemma 4.3, we can show that $U(\cdot)/\varphi(\cdot)$ can be written as the lower envelope of a family of nonnegative F–concave functions, i.e., it is nonnegative and F–concave. To show that $U(\cdot)$ majorizes $h(\cdot)$, assume for a moment that

$$\theta \triangleq \max_{x\in[c,d]}\left(\frac{h(x)}{\varphi(x)} - \frac{U(x)}{\varphi(x)}\right) > 0. \tag{4.9}$$

Since θ is attained at some $x_0 \in [c,d]$, and $[U(\cdot)/\varphi(\cdot)]+\theta$ is a nonnegative, F–concave majorant of $h(\cdot)/\varphi(\cdot)$, Proposition 4.2 implies $h(x_0)/\varphi(x_0) = [U(x_0)/\varphi(x_0)] + \theta \geq V(x_0)/\varphi(x_0) \geq h(x_0)/\varphi(x_0)$; equivalently $x_0 \in \mathbf{\Gamma}$, and $U(x_0) = h(x_0)$, thus $\theta = 0$, contradiction to (4.9). Therefore $U(\cdot) \geq h(\cdot)$ on $[c,d]$, as claimed. □

Remark 4.1. Let B be a one–dimensional standard Brownian motion in $[F(c),F(d)]$ with absorbing boundaries, and W, H be defined as in Proposition 4.3. From Proposition 3.2 of Section 3, we have

$$W(y) \equiv \sup_{\tau\geq 0}\mathbb{E}_y[H(B_\tau)], \quad y \in [F(c),F(d)]. \tag{4.10}$$

If $h(\cdot)$ is continuous on $[c,d]$, then $H(\cdot)$ will be continuous on the closed bounded interval $[F(c),F(d)]$. Therefore, the optimal stopping problem of (4.10) has an optimal rule $\sigma^* \triangleq \{t \geq 0 : B_t \in \widetilde{\mathbf{\Gamma}}\}$, where $\widetilde{\mathbf{\Gamma}} \triangleq \{y \in [F(c),F(d)] : W(y) = H(y)\}$ is the optimal stopping region of the same problem. Moreover $\mathbf{\Gamma} = F^{-1}\big(\widetilde{\mathbf{\Gamma}}\big)$.

In light of Remark 3.1 and Proposition 4.3, there is essentially only one class of optimal stopping problems for one–dimensional diffusions, namely, *the undiscounted optimal stopping problems for Brownian motion.*

We close this section with the proof of necessity in Proposition 4.1; the proof of Proposition 4.2 follows along lines similar to Proposition 3.2.

PROOF OF PROPOSITION 4.1. To prove necessity, suppose $U(\cdot)$ is nonnegative and $U(\cdot)/\varphi(\cdot)$ is F- concave on $[c,d]$. As in the proof of Proposition 3.1, thanks to the strong Markov property of X and the optional sampling theorem for nonnegative supermartingales, it is enough to prove that

$$U(x) \geq \mathbb{E}_x[e^{-\beta(\rho\wedge t)}U(X_{\rho\wedge t})], \quad x \in [c,d],\ t \geq 0, \tag{4.11}$$

where $\rho \triangleq \inf\{t \geq 0 : X_t \notin (c,d)\}$. Clearly, this holds for $x = c$ and $x = d$.

Next fix any $x \in (c,d)$. Since $U(\cdot)/\varphi(\cdot)$ is F–concave on $[c,d]$, there exists an affine transformation $L(\cdot) \triangleq c_1F(\cdot)+c_2$ of the function $F(\cdot)$ on $[c,d]$, such that $L(\cdot) \geq U(\cdot)/\varphi(\cdot)$ and $L(x) = U(x)/\varphi(x)$. Now observe that

$$\begin{aligned}\mathbb{E}_x[e^{-\beta(\rho\wedge t)}U(X_{\rho\wedge t})] &\leq \mathbb{E}_x[e^{\beta(\rho\wedge t)}\varphi(X_{\rho\wedge t})L(X_{\rho\wedge t})] = \mathbb{E}_x\left[e^{-\beta(\rho\wedge t)}\varphi(X_{\rho\wedge t})\big(c_1F(X_{\rho\wedge t})+c_2\big)\right]\\ &= c_1\mathbb{E}_x[e^{-\beta(\rho\wedge t)}\psi(X_{\rho\wedge t})] + c_2\mathbb{E}_x[e^{-\beta(\rho\wedge t)}\varphi(X_{\rho\wedge t})], \quad \forall\, t \geq 0.\end{aligned}$$

Because $\psi(\cdot)$ is of class $C^2[c,d]$, we can apply Itô's Rule to $e^{-\beta t}\psi(X_t)$; the stochastic integral is a square-integrable martingale, since its quadratic variation process is integrable, and because $\mathcal{A}\psi = \beta\psi$ on (c,d) , we obtain

$$\mathbb{E}_x[e^{-\beta(\rho\wedge t)}\psi(X_{\rho\wedge t})] = \psi(x) + \mathbb{E}_x\left[\int_0^{\rho\wedge t} e^{-\beta s}(\mathcal{A}\psi - \beta\psi)(X_s)ds\right] = \psi(x), \qquad \forall\, t \geq 0.$$

Similarly, $\mathbb{E}_x[e^{-\beta(\rho\wedge t)}\varphi(X_{\rho\wedge t})] = \varphi(x)$, whence $\mathbb{E}_x[e^{-\beta(\rho\wedge t)}U(X_{\rho\wedge t})] \leq c_1\psi(x)+c_2\varphi(x) = \varphi(x)L(x) = U(x)$. This proves (4.11). □

5. Boundaries and Optimal Stopping

In Section 3 and 4, we assumed that the process X is allowed to diffuse in a closed and bounded interval, and is absorbed when it reaches either one of the boundaries.

There are many other interesting cases, where the state space may not be compact, or the behavior of the process different near the boundaries.

It is always possible to show that the value function $V(\cdot)$ must satisfy the properties of Proposition 3.2 or Proposition 4.2. Additional necessary conditions on $V(\cdot)$ appear, if one or more boundaries are regular reflecting (for example, the value function $V(\cdot)$ for the undiscounted problem of Section 3 should be non–increasing if c is reflecting, non–decreasing if d is reflecting).

The challenge is to show that $V(\cdot)$ is the smallest function with these necessary conditions. Proposition 3.1 and Proposition 4.1 meet this challenge when the boundaries are absorbing. Their proofs illustrate the key tools. Observe that the local martingales, $S(X_t)$ and the constant 1 of Section 3, and $e^{-\beta t}\psi(X_t)$ and $e^{-\beta t}\varphi(X_t)$ of Section 4, are fundamental in the proofs of sufficiency.

Typically, the concavity of the appropriate quotient of some nonnegative function $U(\cdot)$ with respect to a quotient of the monotone fundamental solutions $\psi(\cdot)$, $\varphi(\cdot)$ of $\mathcal{A}u = \beta u$, as in (2.5) , will imply that $U(\cdot)$ is β–excessive. The main tools in this effort are Itô's rule, the localization of local martingales, the lower semi–continuity of $U(\cdot)$ (usually implied by concavity of some sort), and Fatou's Lemma. Different boundary conditions may necessitate additional care to complete the proof of superharmonicity.

We shall not attempt here to formulate a general theorem that covers all cases. Rather, we shall state and prove in this section, the key propositions for a diffusion process with absorbing and/or natural boundaries. We shall illustrate how the propositions look like, and what additional tools we may need, to overcome potential difficulties with the boundaries.

5.1. **Left–boundary is absorbing, right–boundary is natural.** Suppose the right–boundary $b \le \infty$ of the state–space $\mathcal{I}$ of the diffusion process is natural. Let $c \in \text{int}(\mathcal{I})$. Note that the process, starting in (c, b), reaches c in finite time with positive probability. Consider the stopped process X, which starts in $[c, b)$ and is stopped when it reaches c. Finally, recall the functions $\psi(\cdot)$ and $\varphi(\cdot)$ of (2.5) for some constant $\beta > 0$. Since $c \in \text{int}(\mathcal{I})$, we have $0 < \psi(c) < \infty$, $0 < \varphi(c) < \infty$. Because b is natural, we have $\psi(b-) = \infty$ and $\varphi(b-) = 0$. Let the reward function $h : [c, b) \to \mathbb{R}$ be bounded on compact subsets, and define

$$V(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}_x[e^{-\beta\tau} h(X_\tau)], \quad x \in [c, b).$$

Let $(b_n)_{n\geq 1} \subset [c,b)$ be an increasing sequence such that $b_n \to b$ as $n\to\infty$. Define the stopping times

$$\sigma_n \triangleq \inf\{t\geq 0 : X_t \notin (c,b_n)\},\ n\geq 1;\quad \text{and}\quad \sigma \triangleq \inf\{t\geq 0 : X_t\notin (c,b)\}. \tag{5.1}$$

Note that $\sigma_n \uparrow \sigma$ as $n\to\infty$. Since b is a natural boundary, we in fact have $\sigma = \inf\{t\geq 0 : X_t = c\}$ almost surely. We can now state and prove the key

Proposition 5.1. *For a function $U : [c,b) \to [0,+\infty)$, $U(\cdot)/\psi(\cdot)$ is G–concave on $[c,b)$ if and only if $U(x) \geq \mathbb{E}_x[e^{-\beta\tau}U(X_\tau)]$, holds for every $x\in[c,b)$ and $\tau\in\mathcal{S}$.*

Proof. Sufficiency follows from Lemma 4.3 when we let τ be 0, ∞, and $\tau_l \wedge \tau_r$, for every choice of $x\in[l,r]\subset[c,b)$. For the necessity, we only have to show (as in the proof of Proposition 4.1) that

$$U(x) \geq \mathbb{E}_x[e^{-\beta t}U(X_t)],\quad x\in[c,b),\ t\geq 0. \tag{5.2}$$

And as in the proof of Proposition 4.1, we first prove a simpler version of (5.2), namely

$$U(x) \geq \mathbb{E}_x[e^{-\beta(\sigma\wedge t)}U(X_{\sigma\wedge t})],\quad x\in[c,b),\ t\geq 0. \tag{5.3}$$

The main reason was that the behavior of the process up to the time σ of reaching the boundaries, is completely determined by its infinitesimal generator $\mathcal{A}$. We can therefore use Itô's rule without worrying about what happens after the process reaches the boundaries. In the notation of (5.1), we have

$$U(x) \geq \mathbb{E}_x[e^{-\beta(\sigma_n\wedge t)}U(X_{\sigma_n\wedge t})],\quad x\in[c,b),\ t\geq 0,\ n\geq 1. \tag{5.4}$$

This is obvious, in fact as equality, for $x\notin(c,b_n)$. For $x\in(c,b_n)$, $X_{\sigma_n\wedge t}$ lives in the closed bounded interval $[c,b_n]$ contained in the interior of $\mathcal{I}$; and c and b_n are absorbing for $\{X_{\sigma_n\wedge t};\ t\geq 0\}$. An argument similar to that in the proof of Proposition 4.1, completes the proof of (5.4).

Since $G(\cdot)$ is continuous on $[c,b)$, and $U(\cdot)/\psi(\cdot)$ is G–concave on $[c,b)$, Proposition 2.4 implies that U is lower semi–continuous on $[c,b)$, i.e., $\liminf_{y\to x} U(y)\geq U(x)$, for every $x\in[c,b)$. Because $\sigma_n\wedge t\to\sigma\wedge t$ and $X_{\sigma_n\wedge t}\to X_{\sigma\wedge t}$, as $n\to\infty$, we have

$$\begin{aligned}\mathbb{E}_x[e^{-\beta(\sigma\wedge t)}U(X_{\sigma\wedge t})] &\leq \mathbb{E}_x\big[\liminf_{n\to\infty} e^{-\beta(\sigma_n\wedge t)}U(X_{\sigma_n\wedge t})\big]\\ &\leq \liminf_{n\to\infty}\mathbb{E}_x\big[e^{-\beta(\sigma_n\wedge t)}U(X_{\sigma_n\wedge t})\big]\leq U(x),\end{aligned}$$

from lower semi–continuity, nonnegativity, and (5.4). This proves (5.3). Finally, since c is absorbing, and $\sigma \equiv \inf\{t \geq 0 : X_t = c\}$, we have $X_t = X_\sigma = c$ on $\{t \geq \sigma\}$. Therefore, (5.2) follows from (5.3) as in

$$\mathbb{E}_x[e^{-\beta t}U(X_t)] = \mathbb{E}_x[e^{-\beta t}U(X_{\sigma\wedge t})] \leq \mathbb{E}_x[e^{-\beta(\sigma\wedge t)}U(X_{\sigma\wedge t})] \leq U(x), \quad x \in [c,b),\ t \geq 0,$$

and the proof is complete. □

We shall investigate next, conditions, under which the value–function $V(\cdot)$ is real–valued. It turns out that this is determined by the quantity

$$\ell_b \triangleq \limsup_{x\to b} \frac{h^+(x)}{\psi(x)} \in [0,+\infty], \tag{5.5}$$

where $h^+(\cdot) \triangleq \max\{0, h(\cdot)\}$ on $[c,b)$.

We shall first show that $V(x) = +\infty$ for every $x \in (c,b)$, if $\ell_b = +\infty$. To this end, fix any $x \in (c,b)$. Let $(r_n)_{n\in\mathbb{N}} \subset (x,b)$ be any strictly increasing sequence with limit b. Define the stopping times $\tau_{r_n} \triangleq \inf\{t \geq 0 : X_t \geq r_n\}$, $n \geq 1$. Lemma 4.3 implies

$$V(x) \geq \mathbb{E}_x[e^{-\beta\tau_{r_n}}h(X_{\tau_{r_n}})] = \psi(x)\frac{h(r_n)}{\psi(r_n)} \cdot \frac{G(x)-G(c)}{G(r_n)-G(c)}, \quad n \geq 1.$$

On the other hand, since $\tau \equiv +\infty$ is also a stopping time, we also have $V \geq 0$. Therefore

$$\frac{V(x)}{\psi(x)} \geq 0 \vee \left(\frac{h(r_n)}{\psi(r_n)} \cdot \frac{G(x)-G(c)}{G(r_n)-G(c)}\right) = \frac{h^+(r_n)}{\psi(r_n)} \cdot \frac{G(x)-G(c)}{G(r_n)-G(c)}, \quad n \geq 1. \tag{5.6}$$

Remember that G is strictly increasing and negative (i.e., bounded from above). Therefore $G(b-)$ exists, and $-\infty < G(c) < G(b-) \leq 0$. Furthermore since $x > c$, we have $G(x) - G(c) > 0$. By taking the limit supremum of both sides in (5.6) as $n \to +\infty$, we find

$$\frac{V(x)}{\psi(x)} \geq \limsup_{n\to+\infty} \frac{h^+(r_n)}{\psi(r_n)} \cdot \frac{G(x)-G(c)}{G(r_n)-G(c)} = \ell_b \cdot \frac{G(x)-G(c)}{G(b-)-G(c)} = +\infty.$$

Since $x \in (c,b)$ was arbitrary, this proves that $V(x) = +\infty$ for all $x \in (c,b)$, if ℓ_b of (5.5) is equal to $+\infty$.

Suppose now that ℓ_b is finite. We shall show that $\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)]$ is well–defined in this case for every stopping time τ, and that $V(\cdot)$ is finite on $[c,b)$. Since $\ell_b < \infty$, there exists some $b_0 \in (c,b)$ such that $h^+(x) < (1+\ell_b)\psi(x)$, for every $x \in (b_0,b)$.

Since $h(\cdot)$ is bounded on the closed and bounded interval $[c, b_0]$, we conclude that there exists some finite constant $K > 0$ such that

$$h^+(x) \le K\psi(x), \quad \text{for all } x \in [c, b). \tag{5.7}$$

Now read Proposition 5.1 with $U \triangleq \psi$, and conclude that

$$\psi(x) \ge \mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)], \quad \forall\, x \in [c, b),\ \forall\, \tau \in \mathcal{S}. \tag{5.8}$$

This and (5.7) lead to $K\psi(x) \ge \mathbb{E}_x\big[e^{-\beta\tau}h^+(X_\tau)\big]$, for every $x \in [c, b)$ and every $\tau \in \mathcal{S}$. Thus $\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)]$ is well–defined (i.e., expectation exists) for every stopping time τ, and $K\psi(x) \ge \mathbb{E}_x\big[e^{-\beta\tau}h^+(X_\tau)\big] \ge \mathbb{E}_x\big[e^{-\beta\tau}h(X_\tau)\big]$, for every $x \in [c, b)$ and stopping time τ, which means

$$0 \le V(x) \le K\psi(x) \tag{5.9}$$

i.e., $V(x)$ is finite for every $x \in [c, b)$. The following result has been proved (see Beibel and Lerche [2, Theorem 1] for a conclusion similar to Propositions 5.2 and 5.10. See also Beilbel and Lerche [1]).

Proposition 5.2. *We have either $V \equiv +\infty$ in (c, d), or $V(x) < +\infty$ for all $x \in [c, b)$. Moreover, $V(x) < +\infty$ for every $x \in [c, b)$ if and only if ℓ_b of (5.5) is finite.*

In the remainder of this Subsection, we shall assume that

$$\text{the quantity } \ell_b \text{ of (5.5) is finite,} \tag{5.10}$$

so that $V(\cdot)$ is real–valued. We shall investigate the properties of $V(\cdot)$, and describe how to find it. The main result is as follows; its proof is almost identical to the proof of Proposition 4.2, with the obvious changes, such as the use of Proposition 5.1 instead of Proposition 4.1.

Proposition 5.3. *$V(\cdot)$ is the smallest nonnegative majorant of $h(\cdot)$ on $[c, b)$ such that $V(\cdot)/\psi(\cdot)$ is G–concave on $[c, b)$.*

We shall continue our discussion by first relating ℓ_b of (5.5) to $V(\cdot)$ as in Proposition 5.4. Since $V(\cdot)/\psi(\cdot)$ is G–concave, the limit $\lim_{x\uparrow b} V(x)/\psi(x)$ exists, and (5.9) implies that this limit is finite. Since $V(\cdot)$ moreover majorizes $h^+(\cdot)$, we have

$$\ell_b = \limsup_{x\uparrow b} \frac{h^+(x)}{\psi(x)} \le \lim_{x\uparrow b} \frac{V(x)}{\psi(x)} < +\infty. \tag{5.11}$$

Proposition 5.4. *If $h(\cdot)$ defined and bounded on compact subintervals of $[c,b)$, and if (5.10) holds, then*

$$\lim_{x\uparrow b}\frac{V(x)}{\psi(x)}=\ell_b.$$

Proof. Fix any arbitrarily small $\varepsilon>0$, and note that (5.10) implies that the existence of some $l\in(c,b)$ such that

$$y\in[l,b)\implies h(y)\le h^+(y)\le(\ell_b+\varepsilon)\psi(y). \tag{5.12}$$

For every $x\in(l,b)$ and arbitrary stopping time $\tau\in\mathcal{S}$, we have $\{X_\tau\in[c,l)\}\subseteq\{\tau_l<\tau\}$, on $\{X_0=x\}$. Note also that the strong Markov property of X and (5.8) imply that $e^{-\beta t}\psi(X_t)$ is a nonnegative supermartingale. Consequently,

$$\begin{aligned}
\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)]&=\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)1_{\{X_\tau\in[c,l)\}}]+\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)1_{\{X_\tau\in(l,b)\}}]\\
&\le K\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)1_{\{X_\tau\in[c,l)\}}]+(\ell_b+\varepsilon)\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)1_{\{X_\tau\in(l,b)\}}]\\
&\le K\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)1_{\{\tau_l<\tau\}}]+(\ell_b+\varepsilon)\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)]\\
&\le K\mathbb{E}_x[e^{-\beta\tau_l}\psi(X_{\tau_l})1_{\{\tau_l<\infty\}}]+(\ell_b+\varepsilon)\psi(x)\\
&=K\psi(l)\mathbb{E}_x[e^{-\beta\tau_l}]+(\ell_b+\varepsilon)\psi(x)\\
&\le K\psi(x)\mathbb{E}_x[e^{-\beta\tau_l}]+(\ell_b+\varepsilon)\psi(x)=K\psi(x)\frac{\varphi(x)}{\varphi(l)}+(\ell_b+\varepsilon)\psi(x),
\end{aligned}$$

where the right–hand side no longer depends on the stopping time τ. Therefore,

$$\frac{V(x)}{\psi(x)}\le\frac{K}{\varphi(l)}\varphi(x)+\ell_b+\varepsilon,\quad\text{for every } x\in(l,b).$$

By taking limits on both sides as x tends to b, we obtain

$$\lim_{x\uparrow b}\frac{V(x)}{\psi(x)}\le\frac{K}{\varphi(l)}\varphi(b-)+\ell_b+\varepsilon=\ell_b+\varepsilon,$$

since $\varphi(b-)=0$, and let $\varepsilon\downarrow 0$ to conclude $\lim_{x\uparrow b}V(x)/\psi(x)\le\ell_b$. In conjunction with (5.11), this completes the proof. □

Proposition 5.5. *Let $W:[G(c),0]\to\mathbb{R}$ be the smallest nonnegative majorant of the function $H:[G(c),0]\to\mathbb{R}$, given by*

$$H(y)\triangleq\begin{cases}\dfrac{h\big(G^{-1}(y)\big)}{\psi\big(G^{-1}(y)\big)}, & \textit{if}\quad y\in[G(c),0),\\[2mm] \ell_b, & \textit{if}\quad y=0.\end{cases} \tag{5.13}$$

Then $V(x) = \psi(x) W\big(G(x)\big)$, *for every* $x \in [c, b)$. *Furthermore,* $W(0) = \ell_b$, *and* W *is continuous at* 0.

Since $G(\cdot)$ is continuous on $[c, b)$ and $V(\cdot)/\psi(\cdot)$ is G–concave, $V(\cdot)/\psi(\cdot)$ is continuous on (c, b) and $V(c)/\psi(c) \le \liminf_{x\downarrow c} V(x)/\psi(x)$. However, $\psi(\cdot)$ itself is continuous on $[c, b)$. Therefore, $V(\cdot)$ is continuous on (c, b) and $V(c) \le \liminf_{x\downarrow c} V(x)$. An argument similar to Dynkin and Yushkevich [8] gives

Proposition 5.6. *If* $h : [c, b) \to \mathbb{R}$ *is continuous, and (5.10) is satisfied, then* $V(\cdot)$ *is continuous on* $[c, b)$.

In the remaining part of the subsection, we shall investigate the existence of an optimal stopping time. Proposition 5.7 shows that this is guaranteed when ℓ_b of (5.5) equals zero. Lemma 5.8 gives necessary and sufficient conditions for the existence of an optimal stopping time, when ℓ_b is positive. Finally, no optimal stopping time exists when ℓ_b equals $+\infty$, since the value function equals $+\infty$ everywhere. As usual, we define

$$\mathbf{\Gamma} \triangleq \{x \in [c, b) : V(x) = h(x)\}, \quad \text{and} \quad \tau^* \triangleq \inf\{t \ge 0 : X_t \in \mathbf{\Gamma}\}. \tag{5.14}$$

Remark 5.1. Suppose $W(\cdot)$ and $H(\cdot)$ are functions defined on $[G(c), 0]$ as in Proposition 5.5. If $\widetilde{\mathbf{\Gamma}} \triangleq \{y \in [G(c), 0) : W(y) = H(y)\}$, then $\mathbf{\Gamma} = G^{-1}\big(\widetilde{\mathbf{\Gamma}}\big)$.

Proposition 5.7. *Suppose* $h : [c, b) \to \mathbb{R}$ *is continuous, and* $\ell_b = 0$ *in (5.5). Then* τ^* *of (5.14) is an optimal stopping time.*

Proof. As in the proof of Proposition 4.4, $U(x) \triangleq \mathbb{E}_x[e^{-\beta\tau^*} h(X_{\tau^*})]$, $x \in [c, b)$, is nonnegative, $U(\cdot)/\psi(\cdot)$ is F–concave, and continuous on $[c, b)$. Since $\ell_b = 0$,

$$\theta \triangleq \sup_{x\in[c,b)} \left(\frac{h(x)}{\psi(x)} - \frac{U(x)}{\psi(x)}\right) = \max_{x\in[c,b)} \left(\frac{h(x)}{\psi(x)} - \frac{U(x)}{\psi(x)}\right) \tag{5.15}$$

is attained in $[c, b)$. Now, the same argument as in the proof of Proposition 4.4, shows that $U(\cdot)/\psi(\cdot)$ majorizes $h(\cdot)/\psi(\cdot)$. □

Proposition 5.8. *Suppose* $\ell_b > 0$ *is finite and* $h(\cdot)$ *is continuous. Then* τ^* *of (5.14) is an optimal stopping time if and only if there is no* $l \in [c, b)$ *such that* $(l, b) \subseteq \mathbf{C}$.[1]

[1]This condition is stronger than the statement "for some $l \in [c, b)$, $(l, b) \subseteq \mathbf{\Gamma}$". Indeed, suppose there exists a strictly increasing sequence $b_n \uparrow b$ such that $(b_{n_k}, b_{n_k+1}) \subseteq \mathbf{C}$ for some subsequence $\{b_{n_k}\}_{k\in\mathbb{N}} \subseteq \mathbf{\Gamma}$. The original condition in Lemma 5.8 still holds, but there is no $l \in [c, b)$ such that $(l, b) \subseteq \mathbf{\Gamma}$.

Proof. This last condition guarantees that θ of (5.15) is attained, and the proof of the optimality of τ^* is the same as in Proposition 5.7. Conversely, assume that $(l, b) \subseteq \mathbf{C}$ for some $l \in [c, b)$. Then $\tau_l \le \tau^*$, $\mathbb{P}_x$–a.s., for every $x \in (l, b)$. The optional sampling theorem for nonnegative supermartingales implies

$$V(x) = \mathbb{E}_x[e^{-\beta\tau^*}V(X_{\tau^*})] \le \mathbb{E}_x[e^{-\beta\tau_l}V(X_{\tau_l})] = V(l)\frac{\varphi(x)}{\varphi(l)}, \quad \forall x \in (l, b), \tag{5.16}$$

where the last equality follows from (2.7). Since b is natural, (5.16) and Proposition 5.4 imply

$$\ell_b = \limsup_{x\uparrow b}\frac{V(x)}{\psi(x)} \le \frac{V(l)}{\varphi(l)}\limsup_{x\uparrow b}\frac{\varphi(x)}{\psi(x)} = 0,$$

which contradicts with $\ell_b > 0$. □

5.2. **Both boundaries are natural.** Suppose that both a and b are natural for the process X in $\mathcal{I} = (a, b)$. In other words, we have $\psi(a+) = \varphi(b-) = 0$, $\psi(b-) = \varphi(a+) = +\infty$, and $0 < \psi(x), \varphi(x) < \infty$, for $x \in (a, b)$.

Let the reward function $h : (a, b) \to \mathbb{R}$ be bounded on every compact subset of (a, b). Consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau\in\mathcal{S}}\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)], \quad x \in (a, b).$$

In this subsection, we state the results without proofs; these are similar to the arguments in Subsection 5.1. The key result is as follows.

Proposition 5.9. *For a function $U : (a, b) \to [0, +\infty)$, $U(\cdot)/\varphi(\cdot)$ is F–concave on (a, b) (equivalently, $U(\cdot)/\psi(\cdot)$ is G–concave on (a, b)), if and only if $U(x) \ge \mathbb{E}_x[e^{-\beta\tau}U(X_\tau)]$ for every $x \in (a, b)$ and $\tau \in \mathcal{S}$.*

Proposition 5.10. *We have either $V \equiv +\infty$ in (a, b), or $V(x) < +\infty$ for all $x \in (a, b)$. Moreover, $V(x) < +\infty$ for every $x \in (a, b)$, if and only if*

$$\ell_a \triangleq \limsup_{x\downarrow a}\frac{h^+(x)}{\varphi(x)} \quad \text{and} \quad \ell_b \triangleq \limsup_{x\uparrow b}\frac{h^+(x)}{\psi(x)} \tag{5.17}$$

are both finite.

In the remainder of this Subsection, we shall assume that *the quantities ℓ_a and ℓ_b of (5.17) are finite*. Then $\lim_{x\downarrow a} V(x)/\varphi(x) = \ell_a$, and $\lim_{x\uparrow b} V(x)/\psi(x) = \ell_b$.

Proposition 5.11. *$V(\cdot)$ is the smallest nonnegative majorant of $h(\cdot)$ on (a, b) such that $V(\cdot)/\varphi(\cdot)$ is F–concave (equivalently, $V(\cdot)/\psi(\cdot)$ is G–concave) on (a, b).*

Proposition 5.12. *Let $W : [0,+\infty) \to \mathbb{R}$ and $\widetilde{W} : (-\infty, 0] \to \mathbb{R}$ be the smallest nonnegative concave majorants of*

$$H(y) \triangleq \left\{ \begin{array}{ll} \dfrac{h\big(F^{-1}(y)\big)}{\varphi\big(F^{-1}(y)\big)}, & \text{if} \quad y > 0 \\ \ell_a, & \text{if} \quad y = 0 \end{array} \right\}, \quad \text{and} \quad \widetilde{H}(y) \triangleq \left\{ \begin{array}{ll} \dfrac{h\big(G^{-1}(y)\big)}{\psi\big(G^{-1}(y)\big)}, & \text{if} \quad y < 0 \\ \ell_b, & \text{if} \quad y = 0 \end{array} \right\},$$

respectively. Then $V(x) = \varphi(x) W\big(F(x)\big) = \psi(x)\widetilde{W}\big(G(x)\big)$, for every $x \in (a,b)$. Furthermore, $W(0) = \ell_a$, $\widetilde{W}(0) = \ell_b$, and both $W(\cdot)$ and $\widetilde{W}(\cdot)$ are continuous at 0.

Remark 5.2. Suppose $W(\cdot)$ and $H(\cdot)$ be the functions defined on $[0, +\infty)$ as in Proposition 5.12. Let $\widehat{\mathbf{\Gamma}} \triangleq \{y \in (0,+\infty) : W(y) = H(y)\}$. Then $\mathbf{\Gamma} = F^{-1}\big(\widehat{\mathbf{\Gamma}}\big)$, where $\mathbf{\Gamma}$ is defined as in (5.14).

Proposition 5.13. *$V(\cdot)$ is continuous on (a,b). If $h : (a,b) \to \mathbb{R}$ is continuous, and $\ell_a = \ell_b = 0$, then τ^* of (5.14) is an optimal stopping time.*

Proposition 5.14. *Suppose that ℓ_a, ℓ_b are finite and one of them is strictly positive, and $h(\cdot)$ is continuous. Then τ^* of (5.14) is an optimal stopping time, if and only if*

$$\left\{ \begin{array}{c} \text{there is no } r \in (a,b) \\ \text{such that } (a,r) \subset \mathbf{C} \\ \text{if } \ell_a > 0 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} \text{there is no } l \in (a,b) \\ \text{such that } (l,b) \subset \mathbf{C} \\ \text{if } \ell_b > 0 \end{array} \right\}.$$

6. Examples

In this section we shall illustrate how the results of Sections 3–5 apply to various optimal stopping problems that have been studied in the literature, and to some other ones that are new.

6.1. Pricing an "Up–and–Out" Barrier Put–Option of American Type under the Black–Scholes Model (Karatzas and Wang [12]).

Karatzas and Wang [12] address the pricing problem for an "up–and–out" barrier put–option of American type, by solving the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x \big[e^{-r\tau} (q - S_\tau)^+ \, 1_{\{\tau < \tau_d\}} \big], \quad x \in (0,d) \tag{6.1}$$

using variational inequalities. Here S is the stock price process governed under the risk-neutral measure by

$$dS_t = S_t(r dt + \sigma dB_t), \qquad S_0 = x \in (0,d),$$

where B is standard Brownian motion; and the risk–free interest rate $r > 0$ and the volatility $\sigma > 0$ are constant. The barrier and the strike–price are denoted by $d > 0$ and $q \in (0, d)$, respectively. Moreover $\tau_d \triangleq \inf\{t \geq 0 : S(t) \geq d\}$ is the time when the option becomes "knocked-out". The state space of S is $\mathcal{I} = (0, \infty)$. Since the drift r is positive, 0 is a natural boundary for S, whereas every $c \in \text{int}(\mathcal{I})$ is hit with probability one.

We shall offer here a novel solution for (6.1) by using the techniques of Section 5. For this purpose, denote by $\widetilde{S}_t$ the stopped stock–price process, which starts in $(0, d]$ and is absorbed when it reaches the barrier d.

It is clear from (6.1) that $V(x) \equiv 0$, $x \geq d$. We therefore need to determine V on $(0, d]$. Note that V does not depend on the behavior of stock–price process after it reaches the barrier d, and we can rewrite

$$V(x) = \sup_{\tau \geq 0} \mathbb{E}_x \left[e^{-r\tau} h(\widetilde{S}_\tau) \right], \quad x \in (0, d]$$

where $h(x) \triangleq (q - x)^+$ is the reward function (see Figure 1(a)). The infinitesimal generator $\mathcal{A}$ of S is $\mathcal{A}u(x) \triangleq (\sigma^2/2)x^2 u''(x) + rxu'(x)$, acting on smooth functions $u(\cdot)$. The functions of (2.5) with $\beta = r$, turn out to be

$$\psi(x) = x \quad \text{and} \quad \varphi(x) = x^{-\frac{2r}{\sigma^2}}, \qquad x \in (0, \infty).$$

Observe that $\psi(0+) = 0$, $\varphi(0+) = +\infty$. Thus the left–boundary is natural, and the right–boundary is absorbing. *This is the opposite of the case that we had studied in Subsection 5.1. Therefore, we can obtain relevant results from that section, if we replace* $(\psi(\cdot), G(\cdot), \ell_b)$ *by* $(\varphi(\cdot), F(\cdot), \ell_a)$. The reward function $h(\cdot)$ is continuous on $(0, d]$. Since

$$\ell_0 \triangleq \limsup_{x \to 0} \frac{h^+(x)}{\varphi(x)} = \lim_{x \to 0} \left[(q - x) x^{\frac{2r}{\sigma^2}} \right] = 0,$$

the value function $V(\cdot)$ is finite (Proposition 5.2). Therefore, $V(x) = \varphi(x) W(F(x))$, $x \in (0, d]$ by Proposition 5.5, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^{1 + \frac{2r}{\sigma^2}} \equiv x^\beta, \quad x \in (0, d], \qquad \beta \triangleq 1 + \frac{2r}{\sigma^2} > 1, \tag{6.2}$$

and $W : [0, d^\beta] \to \mathbb{R}$ is the smallest nonnegative concave majorant of

$$H(y) \triangleq \left\{ \begin{array}{ll} \left(\dfrac{h}{\varphi}\right) \circ F^{-1}(y), & y \in (0, d^\beta] \\ \ell_0, & y = 0 \end{array} \right\} = \left\{ \begin{array}{ll} y^{1 - \frac{1}{\beta}} \left(q - y^{\frac{1}{\beta}}\right)^+, & y \in (0, d^\beta] \\ 0, & y = 0 \end{array} \right\}.$$

To identify $W(\cdot)$ explicitly, we shall first sketch $H(\cdot)$. Since $h(\cdot)$ and $\varphi(\cdot)$ are nonnegative, $H(\cdot)$ is also nonnegative. Note that $H \equiv 0$ on $[q^\beta, d^\beta]$. On $(0, q^\beta)$, $H(x) = y^{1-\frac{1}{\beta}}\big(q - y^{\frac{1}{\beta}}\big)$ is twice–continuously differentiable, and

$$H'(y) = q\Big(1 - \frac{1}{\beta}\Big)y^{-\frac{1}{\beta}} - 1, \quad H''(y) = q\,\frac{1-\beta}{\beta^2}\,y^{-(1+\frac{1}{\beta})} < 0, \qquad x \in (0, q^\beta),$$

since $\beta > 1$. Hence H is the strictly concave on $[0, q^\beta]$ (See Figure 1(b)).

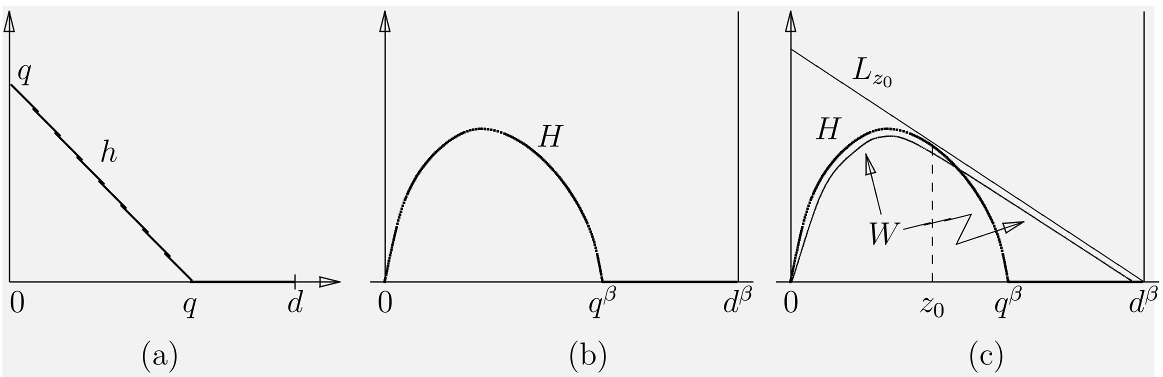

FIGURE 1. Pricing Barrier Option

The strict concavity of H on $[0, q^\beta]$, guarantees the existence of a unique $z_0 \in (0, q^\beta)$ (Figure 1(c)), such that

$$H'(z_0) = \frac{H(d^\beta) - H(z_0)}{d^\beta - z_0} = -\frac{H(z_0)}{d^\beta - z_0}. \tag{6.3}$$

Therefore the straight line $L_{z_0} : [0, d^\beta] \to \mathbb{R}$,

$$L_{z_0}(y) \triangleq H(z_0) + H'(z_0)(y - z_0), \quad y \in [0, d^\beta], \tag{6.4}$$

is tangent to H at z_0 and coincides with the chord expanding between $(z_0, H(z_0))$ and $(d^\beta, H(d^\beta) \equiv 0)$ over the graph of H. Since $H(z_0) > 0$, (6.3) implies that L_{z_0} is decreasing. Therefore $L_{z_0} \geq L_{z_0}(d^\beta) \geq 0$ on $[0, d^\beta]$. It is evident from Figure 1(c) that the smallest nonnegative concave majorant of H on $[0, d^\beta]$ is given by

$$W(y) = \left\{\begin{array}{ll} H(y), & \text{if } y \in [0, z_0] \\ L_{z_0}(y), & \text{if } y \in (z_0, d^\beta] \end{array}\right\} = \left\{\begin{array}{ll} H(y), & \text{if } y \in [0, z_0] \\ H(z_0)\dfrac{d^\beta - y}{d^\beta - z_0}, & \text{if } y \in (z_0, d^\beta] \end{array}\right\}$$

thanks to (6.3) and (6.4). The strict concavity of H on $[0, q^\beta]$ also implies that $\widetilde{\mathbf{C}} \triangleq \{y \in [0, d^\beta] : W(y) > H(y)\} = (z_0, d^\beta)$.

From (6.2), we find $F^{-1}(y) = y^{1/\beta}$, $y \in [0, d^\beta]$. Let $x_0 \triangleq F^{-1}(z_0) = z_0^{1/\beta}$. Then $x_0 \in (0, d)$, and

$$V(x) = \varphi(x)W(F(x)) = \begin{cases} q - x, & 0 \le x \le x_0, \\ (q - x_0) \cdot \dfrac{x}{x_0} \cdot \dfrac{d^{-\beta} - x^{-\beta}}{d^{-\beta} - x_0^{-\beta}}, & x_0 < x \le d. \end{cases} \tag{6.5}$$

Since $\ell_0 = 0$ and h is continuous, the stopping time τ^* of (5.14) is optimal (Proposition 5.7). Because the optimal continuation region becomes $\mathbf{C} \triangleq \{x \in (0, d] : V(x) > h(x)\} = F^{-1}\big(\widetilde{\mathbf{C}}\big) = F^{-1}\big((z_0, d^\beta)\big) = (x_0, d)$ (Remark 5.1), the optimal stopping time becomes $\tau^* = \inf\{t \ge 0 : S_t \notin (x_0, d)\}$. Finally, (6.3) can be rewritten

$$1 + \beta \frac{x_0}{q} = \beta + \left(\frac{x_0}{d}\right)^\beta, \tag{6.6}$$

after some simple algebra using formulae for H, H' and $x_0 \equiv z_0^{1/\beta}$. Compare (6.5) and (6.6) above with (2.18) and (2.19) in Karatzas and Wang [12, pages 263 and 264], respectively.

6.2. **Pricing an "Up–and–Out" Barrier Put–Option of American Type under the Constant–Elasticity–of–Variance (CEV) Model.** We shall look at the same optimal stopping problem of (6.1) by assuming now that the stock price dynamics are described according to the CEV model,

$$dS_t = rS_t dt + \sigma S_t^{1-\alpha} dB_t, \qquad S_0 \in (0, d),$$

for some $\alpha \in (0, 1)$. The infinitesimal generator for this process is $\mathcal{A} = \frac{1}{2}\sigma^2 x^{2(1-\alpha)} \frac{d^2}{dx^2} + rx\frac{d}{dx}$, and the functions of (2.5) with $\beta = r$ are given by

$$\psi(x) = x, \quad \varphi(x) = x \cdot \int_x^{+\infty} \frac{1}{z^2} \exp\left\{-\frac{r}{\alpha\sigma^2} z^{2\alpha}\right\} dz, \qquad x \in (0, +\infty),$$

respectively. Moreover $\psi(0+) = 0$, $\varphi(0+) = 1$ and $\psi(+\infty) = +\infty$, $\varphi(+\infty) = 0$. Therefore 0 is an exit–and–not–entrance boundary, and $+\infty$ is a natural boundary for S. We shall regard 0 as an absorbing boundary (i.e., up on reaching 0, we shall assume that the process remains there forever). We shall also modify the process such that d becomes an absorbing boundary. Therefore, we have our optimal stopping problem in the canonical form of Section 4, with the reward function $h(x) = (q - x)^+$, $x \in [0, d]$.

One can show that the results of Section 4 stay valid when the left–boundary of the state space is an exit–and–not–entrance boundary. According to Proposition 4.3, $V(x) = \psi(x)W(G(x))$, $x \in [0,d]$ with

$$G(x) \triangleq -\frac{\varphi(x)}{\psi(x)} = -\int_x^{+\infty} \frac{1}{u^2} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du, \qquad x \in (0,d], \tag{6.7}$$

and $W : (-\infty, G(d)] \to \mathbb{R}$ $(G(0+) = -\infty)$ is the smallest nonnegative concave majorant of $H : (-\infty, G(d)] \to \mathbb{R}$, given by

$$H(y) \triangleq \left(\frac{h}{\psi} \circ G^{-1}\right)(y) = \left\{ \begin{array}{ll} \left[\left(\frac{q}{x} - 1\right) \circ G^{-1}\right](y), & \text{if } -\infty < y < G(q) \\ 0, & \text{if } G(q) \le y \le 0 \end{array} \right\}. \tag{6.8}$$

Except for $y = G(q)$, H is twice–differentiable on $(-\infty, G(d))$. It can be checked that H is strictly decreasing and strictly concave on $(-\infty, G(q))$. Moreover $H(-\infty) = +\infty$ and $H'(-\infty) = -q$, since $G^{-1}(-\infty) = 0$.

For every $-\infty < y < G(q)$, let $z(y)$ be the point on the y–axis, where the tangent line $L_y(\cdot)$ of $H(\cdot)$ at y intersects the y–axis (cf. Figure 2(a)). Then

(6.9)

$$\begin{aligned} z(y) = y - \frac{H(y)}{H'(y)} &= G(G^{-1}(y)) - \frac{\left[\left(\frac{q}{x} - 1\right) \circ G^{-1}\right](y))}{\left[\left(-q \exp\left\{\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}\right) \circ G^{-1}\right](y)} \\ &= \left[\left(\frac{2r}{\sigma^2}\int_x^{+\infty} u^{2(\alpha-1)} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du - \frac{1}{q}\exp\left\{-\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}\right) \circ G^{-1}\right](y), \end{aligned}$$

where the last equality follows from integration by parts. It is geometrically clear that $z(\cdot)$ is strictly decreasing. Since $G^{-1}(-\infty) = 0$, we have

$$z(-\infty) = \frac{2r}{\sigma^2}\int_0^{+\infty} u^{2(\alpha-1)} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du - \frac{1}{q}$$

Note that $G(q) < z(-\infty) < +\infty$ if $1/2 < \alpha < 1$, and $z(-\infty) = +\infty$ if $0 < \alpha \le 1/2$.

Case I. Suppose first $G(d) < z(-\infty)$ (especially, when $0 < \alpha \le 1/2$). Then there exists a unique $y_0 \in (-\infty, G(q))$ such that $z(y_0) = G(d)$, thanks to the monotonicity and continuity of $z(\cdot)$. In other words, the tangent line $L_{y_0}(\cdot)$ of $H(\cdot)$ at $y = y_0 < G(q)$ intersects y–axis at $y = G(d)$. It is furthermore clear from Figure 2(a) that

$$W(y) = \left\{ \begin{array}{ll} H(y), & \text{if } -\infty < y \le y_0 \\ H(y_0)\dfrac{G(d) - y}{G(d) - y_0}, & \text{if } y_0 < y \le G(d) \end{array} \right\}$$

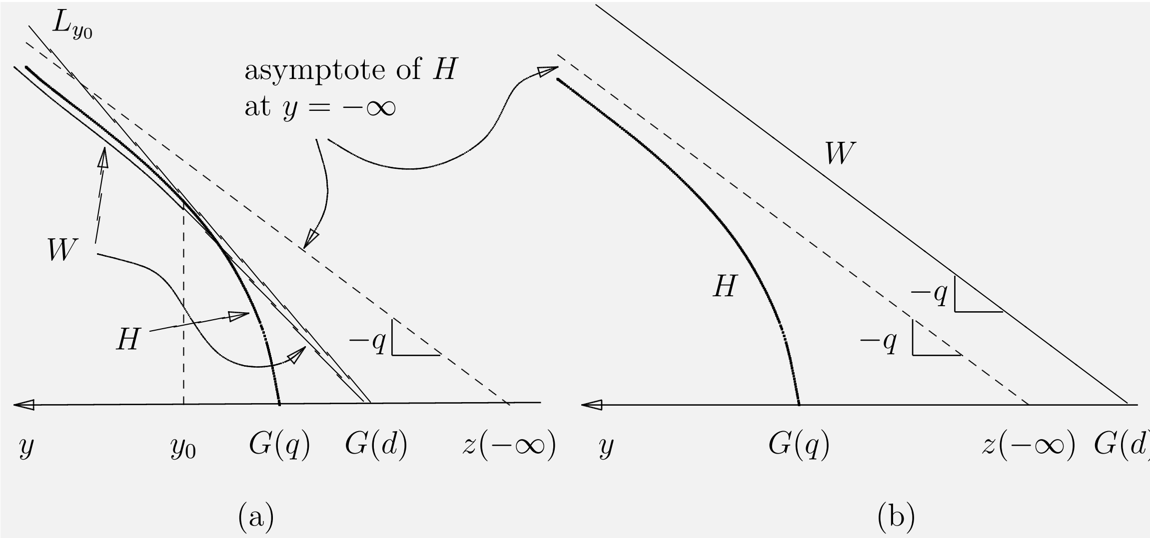

FIGURE 2. (Pricing Barrier Options under the CEV Model) Sketches of the functions H and W of Proposition 4.3, when (a) $G(d) < z(-\infty)$ (for this sketch, we assume that $z(-\infty)$ is finite. However, $z(-\infty) = +\infty$ is also possible, in which case H does not have a linear asymptote), and (b) $G(d) > z(-\infty)$.

is the smallest nonnegative concave majorant of H of (6.8) on $y \in (-\infty, G(d)]$. Define $x_0 \triangleq G^{-1}(y_0)$. According to Proposition 4.3, $V(x) = \psi(x)W(G(x))$, $x \in [0, d]$, i.e.,

$$V(x) = \begin{Bmatrix} q - x, & \text{if } 0 \le x \le x_0 \\ (q - x_0) \cdot \dfrac{x}{x_0} \cdot \dfrac{G(d) - G(x)}{G(d) - G(x_0)}, & \text{if } x_0 < x \le d \end{Bmatrix}.$$

The optimal continuation region becomes $\mathbf{C} = (x_0, d)$, and $\tau^* \triangleq \inf\{t \ge 0 : S_t \notin (x_0, d)\}$ is an optimal stopping time. The relation $z(G(x_0)) = G(d)$, which can be written as

$$\frac{2r}{\sigma^2} \int_{x_0}^{d} u^{2(\alpha-1)} \exp\left\{-\frac{r}{\alpha\sigma^2} u^{2\alpha}\right\} du = \frac{1}{q} \exp\left\{-\frac{r}{\alpha\sigma^2} x_0^{2\alpha}\right\} - \frac{1}{d} \exp\left\{-\frac{r}{\alpha\sigma^2} d^{2\alpha}\right\},$$

determines $x_0 \in (q, d)$ uniquely.

Case II. Suppose now $G(d) > z(-\infty)$ (cf. Figure 2(b)). It is then clear that $W(y) = -q\big[y - G(d)\big]$ is the smallest nonnegative concave majorant of $H(\cdot)$ of (6.8) on $(-\infty, G(d)]$. According to Proposition 4.3, $V(x) = \psi(x)W(G(x)) = -qx\big[G(x) - G(d)\big]$, $x \in [0, d]$, with $V(0) = V(0+) = q$. Furthermore, the stopping time $\tau^* \triangleq \inf\{t \ge 0 : S_t \notin (0, d)\}$ is optimal.

6.3. **American Capped Call Option on Dividend–Paying Assets (Broadie and Detemple [3]).** Let the stock price be driven by

$$dS_t = S_t\big[(r-\delta)dt + \sigma dB_t\big], \quad t \geq 0, \;\; S_0 > 0,$$

with constant $\sigma > 0$, risk–free interest rate $r > 0$ and dividend rate $\delta \geq 0$. Consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x \left[e^{-r\tau}\left(S_\tau \wedge L - K\right)^+\right], \quad x \in (0,+\infty), \tag{6.10}$$

with the reward function $h(x) \triangleq (x \wedge L - K)^+$, $x > 0$. The value function $V(\cdot)$ is the arbitrage–free price of the *perpetual* American capped call option with strike price $K \geq 0$, and the cap $L > K$ on the stock S, which pays dividend at a constant rate δ. We shall reproduce the results of Broadie and Detemple [3] in this subsubsection.

The infinitesimal generator of X coincides with the second–order differential operator $\mathcal{A} \triangleq (\sigma^2/2)x^2 \frac{d^2}{dx^2} + (r-\delta)x\frac{d}{dx}$. Let $\gamma_1 < 0 < \gamma_2$ be the roots of

$$\frac{1}{2}\sigma^2\, x^2 + \left(r - \delta - \frac{\sigma^2}{2}\right)\, x - r = 0.$$

Then the increasing and decreasing solutions of $\mathcal{A}u = ru$ are given by

$$\psi(x) = x^{\gamma_2}, \quad \text{and} \quad \varphi(x) = x^{\gamma_1}, \qquad x > 0,$$

respectively. Both endpoints of the state–space $\mathcal{I} = (0,+\infty)$ of S are natural (Subsection 5.2). Since

$$\ell_0 \triangleq \limsup_{x \downarrow 0} \frac{h^+(x)}{\varphi(x)} = 0, \quad \text{and} \quad \ell_{+\infty} \triangleq \limsup_{x \to +\infty} \frac{h^+(x)}{\psi(x)} = 0,$$

the value function $V(\cdot)$ of (6.10) is finite, and the stopping time τ^* of (5.14) is optimal (Proposition 5.13). Moreover $V(x) = \varphi(x) W(F(x))$, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^\theta, \quad x > 0, \quad \text{and} \quad \theta \triangleq \gamma_2 - \gamma_1 > 0,$$

and $W : [F(0+), F(+\infty)) \to [0,+\infty)$ is the smallest nonnegative concave majorant of $H : [F(0+), F(+\infty)) \to [0,+\infty)$, given by

$$H(y) \triangleq \left(\frac{h}{\varphi}\right)\left(F^{-1}(y)\right) = \left\{\begin{array}{ll} 0, & \text{if } \; 0 \leq y < K^\theta, \\ \left(y^{1/\theta} - K\right) y^{-\gamma_1/\theta}, & \text{if } \; K^\theta \leq y < L^\theta \\ (L-K) y^{-\gamma_1/\theta}, & \text{if } \; y \geq L^\theta, \end{array}\right\}, \tag{6.11}$$

thanks to Proposition 5.12. The function $H(\cdot)$ is nondecreasing on $[0,+\infty)$ and strictly concave on $[L^\theta,+\infty)$. By solving the inequality $H''(y)\le 0$, for $K^\theta\le y\le L^\theta$, we find that

$$H(\cdot)\text{ is }\left\{\begin{array}{ll}\text{convex on} & \big[K^\theta,L^\theta\big]\cap\big[0,(r/\delta)^\theta K^\theta\big]\\ \text{concave on} & \big[K^\theta,L^\theta\big]\cap\big[(r/\delta)^\theta K^\theta,+\infty\big)\end{array}\right\}.$$

It is easy to check that $H(L^\theta)/L^\theta\ge H'(L^\theta+)$ (cf. Figure 3).

Let $\mathcal{L}_z(y)\triangleq y\,H(z)/z$, for every $y\ge 0$ and $z>0$. If $(r/\delta)K\ge L$, then

$$\mathcal{L}_{L^\theta}(y)\ge H(y),\quad y\ge 0, \tag{6.12}$$

(cf. Figure 3(b)). If $(r/\delta)K<L$, then (6.12) holds if and only if

$$\frac{H(L^\theta)}{L^\theta}<H'(L^\theta-)\quad\Longleftrightarrow\quad \gamma_2\le\frac{L}{L-K},$$

(cf. Figure 3(d,f)). If $(r/\delta)K<L$ and $\gamma_2>L/(L-K)$, then the equation $H(z)/z=H'(z)$, $K^\theta<z<L^\theta$ has unique solution, $z_0\triangleq\big[\gamma_2/(\gamma_2-1)\big]^\theta K^\theta>(r/\delta)^\theta K^\theta$, and $\mathcal{L}_{z_0}(y)\ge H(y)$, $y\ge 0$, (cf. Figure 3(c,e)). It is now clear that the smallest nonnegative concave majorant of $H(\cdot)$ is

$$W(y)=\left\{\begin{array}{ll}\mathcal{L}_{z_0\wedge L^\theta}(y), & \text{if }\ 0\le y\le z_0\wedge L^\theta\\ H(y), & \text{if }\ y>z_0\wedge L^\theta\end{array}\right\}$$

in all cases. Finally

$$V(x)=\varphi(x)W(F(x))=\left\{\begin{array}{ll}(x_0\wedge L-K)\left(\dfrac{x}{x_0\wedge L}\right)^{\gamma_2}, & \text{if }\ 0<x\le x_0\wedge L\\ x\wedge L-K, & \text{if }\ x>x_0\wedge L\end{array}\right\},$$

where $x_0\triangleq F^{-1}(z_0)=K\,\gamma_2/(\gamma_2-1)$. The optimal stopping region is $\mathbf{\Gamma}\triangleq\{x:V(x)=h(x)\}=[x_0\wedge L,+\infty)$, and the stopping time $\tau^*\triangleq\inf\{t\ge 0:S_t\in\mathbf{\Gamma}\}=\inf\{t\ge 0:S_t\ge x_0\wedge L\}$ is optimal. Finally, it is easy to check that $\gamma_2=1$ (therefore $x_0=+\infty$) if and only if $\delta=0$.

6.4. **Options for Risk–Averse Investors (Guo and Shepp [9]).** Let X be a geometric Brownian Motion with constant drift $\mu\in\mathbb{R}$ and dispersion $\sigma>0$. Consider the optimal stopping problem

$$V(x)\triangleq\sup_{\tau\ge 0}\ \mathbb{E}_x\big[e^{-r\tau}(l\vee X_\tau)\big],\quad x\in(0,\infty), \tag{6.13}$$

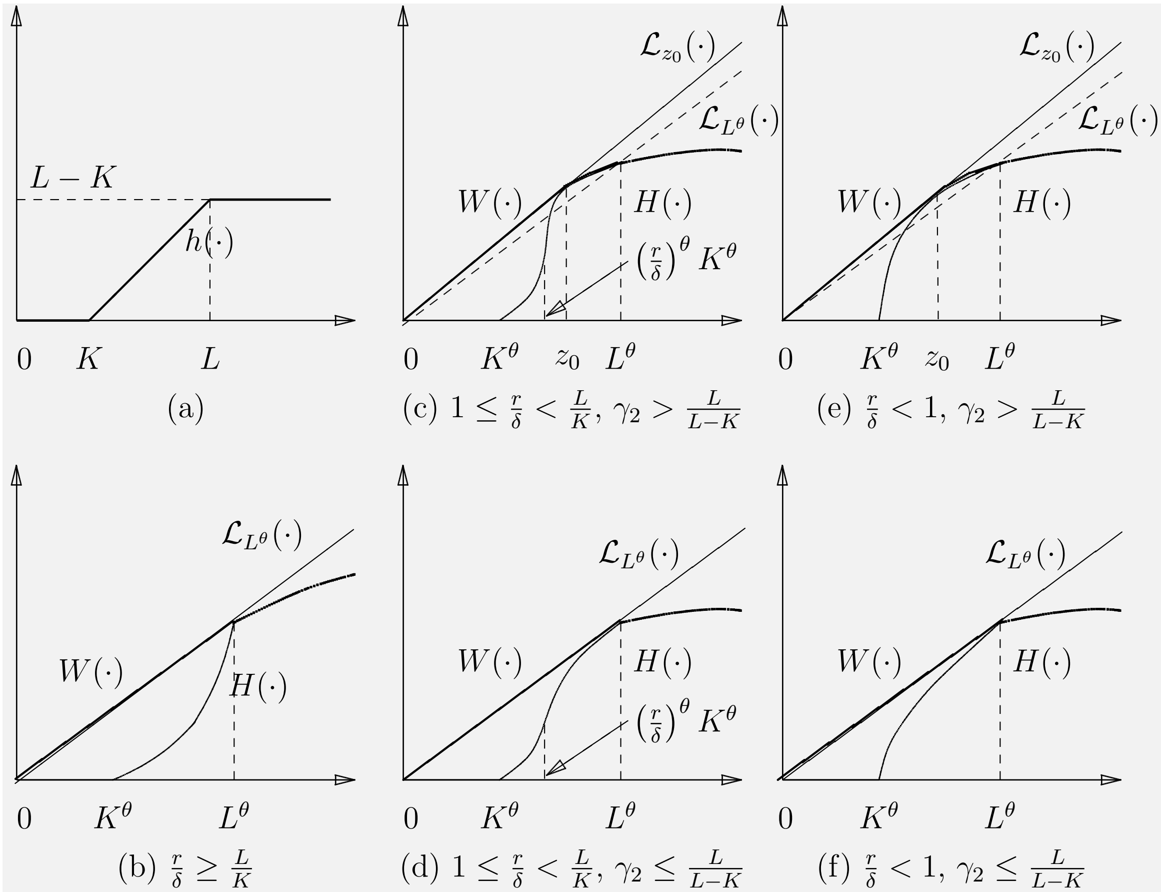

FIGURE 3. (Perpetual American capped call options on dividend–paying assets) Sketches of (a) the reward function $h(\cdot)$, and (b)–(f) the function $H(\cdot)$ of (6.11) and its smallest nonnegative concave majorant $W(\cdot)$.

In cases (b), (d) and (f), the left boundary of the optimal stopping region for the auxiliary optimal stopping problem of (4.10) becomes L^θ, and $W(\cdot)$ does not fit $H(\cdot)$ smoothly at L^θ. In cases (c) and (e), the left boundary of optimal stopping region, namely z_0, is smaller than L^θ, and $W(\cdot)$ fits $H(\cdot)$ smoothly at z_0.

where the reward function is given as $h(x) \triangleq (l \vee x)$, $x \in [0, \infty)$, and l and r positive constants.

Guo and Shepp [9] solve this problem using variational inequalities in order to price exotic options of American type. As it is clear from the reward function, the buyer of the option is guaranteed at least l when the option is exercised (an insurance for risk–averse investors). If r is the riskless interest rate, then the price of the option will be obtained when we choose $\mu = r$. The dynamics of X are given as

$$dX_t = X_t(\mu dt + \sigma dB_t), \qquad X_t = x \in (0,\infty),$$

where B is standard Brownian motion in $\mathbb{R}$. The infinitesimal generator of X coincides with the second–order differential operator $\mathcal{A} = (\sigma^2 x^2/2)(d^2/dx^2) + \mu x(d/dx)$ as it acts on smooth functions. Denote by $\gamma_1, \gamma_0 \triangleq \frac{1}{2}\Big[-\big(\frac{2\mu}{\sigma^2}-1\big) \mp \sqrt{\big(\frac{2\mu}{\sigma^2}-1\big)^2 + \frac{8r}{\sigma^2}}\Big]$, with $\gamma_1 < 0 < \gamma_0$, the roots of the second–order polynomial

$$f(x) \triangleq x^2 + \left(\frac{2\mu}{\sigma^2} - 1\right)x - \frac{2r}{\sigma^2}.$$

The positive increasing and decreasing solutions of $\mathcal{A}u = ru$ are then given as

$$\psi(x) = x^{\gamma_0}, \quad \text{and} \quad \varphi(x) = x^{\gamma_1}, \qquad x \in (0,+\infty),$$

respectively. Observe that both end–points, 0 and $+\infty$, of state space of X are natural, and

$$\ell_0 \triangleq \limsup_{x\to 0} \frac{h^+(x)}{\varphi(x)} = 0, \quad \text{whereas} \qquad \ell_\infty \triangleq \limsup_{x\to+\infty} \frac{h^+(x)}{\psi(x)} = \left\{\begin{array}{ll} +\infty, & \text{if } r<\mu \\ 1, & \text{if } r=\mu \\ 0, & \text{if } r>\mu \end{array}\right\}.$$

Now Proposition 5.10 and 5.13 imply that

$$\left\{\begin{array}{ll} V \equiv +\infty, & \text{if } r<\mu \\ V \text{ is finite, but there is no optimal stopping time,} & \text{if } r=\mu \\ V \text{ is finite, and } \tau^* \text{ of (5.14) is an optimal stopping time,} & \text{if } r>\mu \end{array}\right\}.$$

(Compare with Guo and Shepp[9, Theorem 4 and 5]). There is nothing more to say about the case $r < \mu$. We shall defer the case $r = \mu$ to the next Subsection[2]. We shall study the case $r > \mu$ in the remainder of this Subsection.

According to Proposition 5.12, $V(x) = \varphi(x)W(F(x)) = x^{\gamma_1}W(x^\beta)$, $x \in (0,\infty)$, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^{\gamma_0-\gamma_1} \equiv x^\beta, \quad x \in (0,\infty), \qquad \beta \triangleq \gamma_0 - \gamma_1,$$

[2]In Subsection 6.5, we discuss a slightly different and more interesting problem, of essentially the same difficulty as the problem with $r = \mu$.

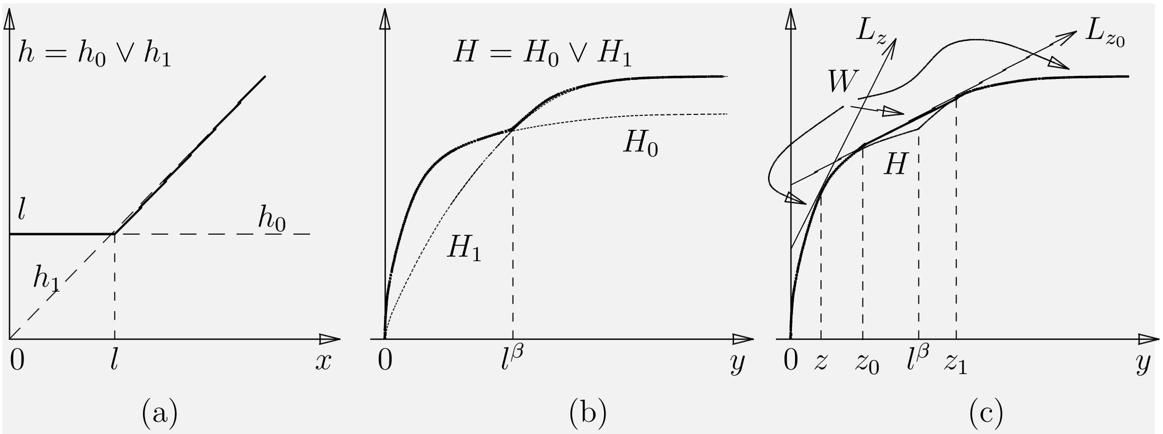

FIGURE 4. Options for risk–averse investors

and $W : [0,\infty) \to \mathbb{R}$ is the smallest nonnegative concave majorant of

$$H(y) \triangleq \left\{ \begin{array}{ll} \dfrac{h\big(F^{-1}(y)\big)}{\varphi\big(F^{-1}(y)\big)}, & \text{if} \quad y \in (0,+\infty) \\ \ell_0, & \text{if} \quad y = 0 \end{array} \right\} = \left\{ \begin{array}{ll} H_0(y) \equiv l y^{-\frac{\gamma_1}{\beta}}, & \text{if} \quad 0 \leq y < l^\beta \\ H_1(y) \equiv y^{\frac{1-\gamma_1}{\beta}}, & \text{if} \quad y \geq l^\beta \end{array} \right\}.$$

In order to find $W(\cdot)$, we shall determine the convexities and the concavities of $H(\cdot)$, which is in fact the maximum of the concave functions $H_0(\cdot)$ and $H_1(\cdot)$, with $H_0(\cdot) > H_1(\cdot)$ on $[0, l^\beta)$ and $H_0(\cdot) < H_1(\cdot)$ on (l^β, ∞). The function $H(\cdot)$ is strictly increasing and continuously differentiable on $(0,\infty)\backslash\{l^\beta\}$ (Figure 4(b)). There exist unique $z_0 \in (0, l^\beta)$ and unique $z_1 \in (l^\beta, \infty)$ (Figure 4(c)), such that

$$H'(z_0) = \frac{H(z_1) - H(z_0)}{z_1 - z_0} = H'(z_1). \tag{6.14}$$

Since both H_0 and H_1 are concave, the line-segment $L_{z_0}(y) \triangleq H(z_0) + H'(z_0)(y - z_0)$, $y \in (0,\infty)$, which is tangent to both H_0 and H_1, majorizes H on $[0,+\infty)$. The smallest nonnegative concave majorant W of H on $[0,\infty)$ is finally given by (cf. Figure 4(c))

$$W(y) = \begin{cases} H(y), & y \in [0, z_0] \cup [z_1, \infty), \\ L_{z_0}(y), & y \in (z_0, z_1). \end{cases}$$

By solving two equations in (6.14) simultaneously, we obtain

$$z_0 = l^\beta \left(\frac{\gamma_1}{\gamma_1 - 1}\right)^{1-\gamma_1} \left(\frac{\gamma_0 - 1}{\gamma_0}\right)^{1-\gamma_0} \quad \text{and} \quad z_1 = l^\beta \left(\frac{\gamma_1}{\gamma_1 - 1}\right)^{-\gamma_1} \left(\frac{\gamma_0 - 1}{\gamma_0}\right)^{-\gamma_0}, \tag{6.15}$$

and, if $x_0 \triangleq F^{-1}(z_0) = z_0^{1/\beta}$ and $x_1 \triangleq F^{-1}(z_1) = z_1^{1/\beta}$, then Proposition 5.12 implies

$$(6.16)\quad V(x) = \varphi(x)W(F(x)) = \begin{cases} l, & \text{if } 0 < x \le x_0, \\ \dfrac{l}{\beta}\left[\gamma_0\left(\dfrac{x}{x_0}\right)^{\gamma_1} - \gamma_1\left(\dfrac{x}{x_0}\right)^{\gamma_0}\right], & \text{if } x_0 < x < x_1, \\ x, & \text{if } x \ge x_1. \end{cases}$$

Moreover, since $\widetilde{\mathbf{C}} \triangleq \{y \in (0,\infty) : W(y) > H(y)\} = (z_0, z_1)$, $\mathbf{C} \triangleq \{x \in (0,\infty) : V(x) > h(x)\} = F^{-1}(\widetilde{\mathbf{C}}) = F^{-1}((z_0,z_1)) = (x_0,x_1)$. Hence $\tau^* \triangleq \inf\{t \ge 0 : X_t \notin (x_0,x_1)\}$ is an optimal stopping rule by Proposition 5.13. Compare (6.16) with (19) in Guo and Shepp [9] (al and bl of Guo and Shepp [9] correspond to x_0 and x_1 in our calculations).

6.5. **Another Exotic Option of Guo and Shepp** [9]**.** The following example is quite instructive, since it provides the opportunity to illustrate new ways for finding the function $W(\cdot)$ of Proposition 5.12. It serves to sharpen the intuition about different forms of smallest nonnegative concave majorants, and how they arise.

Let X be a geometric Brownian motion with constant drift $r > 0$ and dispersion $\sigma > 0$. Guo and Shepp [9] study the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \ge 0} \mathbb{E}_x\left[e^{-r\tau}\left([l \vee X_\tau] - K\right)^+\right], \quad x \in (0,\infty),$$

where l and K are positive constants and $l > K$. The reward function $h(x) \triangleq \left([l \vee x] - K\right)^+$ can be seen as the payoff of some exotic option of American type. The riskless interest rate is $r > 0$, and $K > 0$ is the strike–price of the option. The buyer of the option will be guaranteed to be paid at least $l - K > 0$ at the time of exercise. The value function $V(\cdot)$ is the maximum expected discounted payoff that the buyer can earn. If exists, we want to determine the best time to exercise the option. See Guo and Shepp [9] for more discussion about the option's properties.

As in the first subsection, the generator of X is $\mathcal{A} = (\sigma^2x^2/2)\,(d^2/dx^2) + rx\,(d/dx)$, and the functions of (2.5) with $\beta = r$ are given by

$$\psi(x) = x \quad \text{and} \quad \varphi(x) = x^{-\frac{2r}{\sigma^2}}, \quad x \in (0,\infty).$$

Both boundaries are natural, $h(\cdot)$ is continuous in $(0,\infty)$, and

$$\ell_0 \triangleq \limsup_{x\to 0} \frac{h^+(x)}{\varphi(x)} = 0 \quad \text{and} \quad \ell_\infty \triangleq \limsup_{x\to\infty} \frac{h^+(x)}{\psi(x)} = 1.$$

Since h is bounded on every compact subset of $(0,\infty)$, and both ℓ_0 and ℓ_∞ are finite, V is finite by Proposition 5.10. Proposition 5.12 implies $V(x) = \varphi(x)W(F(x))$, $x \in (0,\infty)$, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^\beta,\ x\in(0,\infty), \quad \text{with} \quad \beta \triangleq 1+\frac{2r}{\sigma^2} > 1,$$

and $W : [0,\infty) \to \mathbb{R}$ is the smallest nonnegative concave majorant of

$$H(y) \triangleq \left\{ \begin{array}{ll} \dfrac{h}{\varphi}\circ F^{-1}(y), & y\in(0,\infty) \\ \ell_0, & y=0. \end{array} \right\} = \left\{ \begin{array}{ll} (l-K)y^{1-1/\beta}, & 0\le y\le l^\beta \\ (y^{1/\beta}-K)y^{1-1/\beta}, & y>l^\beta \end{array} \right\}.$$

In order to find W explicitly, we shall identify the concavities of H. Note that $H'>0$ and $H''<0$ on $(0,l^\beta)$, i.e., H is strictly increasing and strictly concave on $[0,l^\beta]$; furthermore $H'(0+)=+\infty$. On the other hand, $H''>0$, i.e., H is strictly convex, on $(l^\beta,+\infty)$. We also have that H is increasing on $(l^\beta,+\infty)$. One important observation which is key to our investigation of W is that H' is bounded, and asymptotically grows to one:

$$0 < H'(l^\beta-) < H'(y) < 1,\ y>l^\beta; \quad \text{and} \quad \lim_{y\to+\infty} H'(y) = 1.$$

Figure 5(b) illustrates a sketch of H. Since $H'(0+)=+\infty$ and $H'(l^\beta-)<1$, the

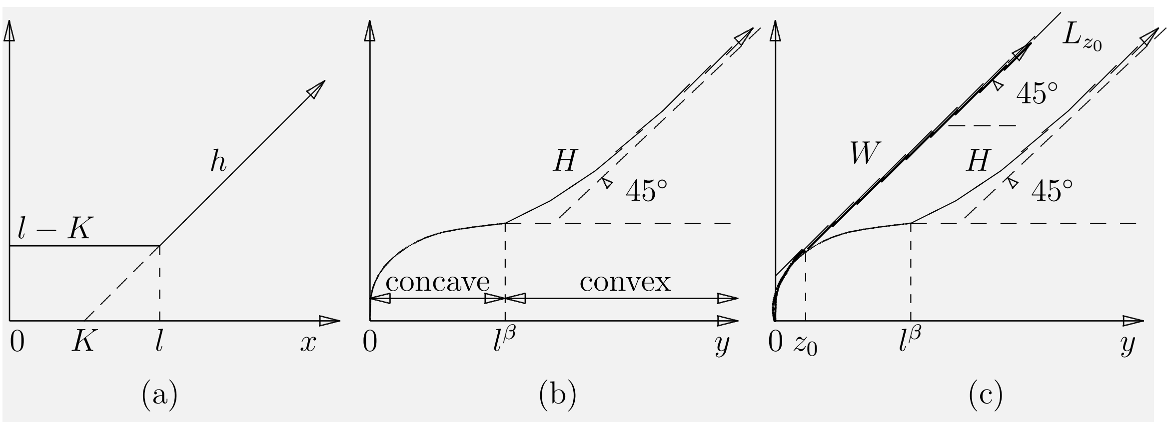

FIGURE 5. Another Exotic Option

continuity of H' and the strict concavity of H in $(0,l^\beta)$ imply that there exists a unique $z_0\in(0,l^\beta)$ such that $H'(z_0)=1$. If $L_{z_0}(y) \triangleq H(z_0)+H'(z_0)(y-z_0) = H(z_0)+y-z_0$,

$y \in [0,\infty)$, is the straight line, tangent to H at z_0 (cf. Figure 5(c)), then

$$W(y) = \begin{Bmatrix} H(y), & 0 \le y \le z_0, \\ L_{z_0}(y), & y > z_0. \end{Bmatrix}, \qquad y \in [0,\infty),$$

and

(6.17)
$$V(x) = \varphi(x) W(F(x)) = \begin{cases} l - K, & 0 < x < x_0, \\ (l-K)\left[\left(1-\dfrac{1}{\beta}\right)\dfrac{x}{x_0} + \dfrac{1}{\beta}\left(\dfrac{x}{x_0}\right)^{1-\beta}\right], & x > x_0, \end{cases}$$

where $x_0 \triangleq F^{-1}(z_0)$ satisfies $x_0 = z_0^{1/\beta} = (1-1/\beta)\big(l-K\big)$. Compare (6.17) with Corollary 3 in Guo and Shepp [9] (In their notation $\gamma_0 = 1$, $\gamma_0 - \gamma_1 = \beta$, $l^* = x_0$.) Finally, there is no optimal stopping time, since $\ell_\infty = 1 > 0$ and $(l,+\infty) \subseteq \mathbf{C} \triangleq \{x : V(x) > h(x)\}$ (Proposition 5.14).

6.6. **An Example of H. Taylor** [17]**.** Let X be one–dimensional Brownian motion with constant drift $\mu \le 0$ and variance coefficient $\sigma^2 = 1$ in $\mathbb{R}$. Taylor [17, Example 1] studies the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \ge 0} \mathbb{E}_x[e^{-\beta\tau}(X_\tau)^+], \qquad x \in \mathbb{R},$$

where the discounting rate $\beta > 0$ is constant. He guesses the value function and verifies that his guess is indeed the nonnegative β–excessive majorant of the reward function $h(x) \triangleq x^+ = \max\{0,x\}$, $x \in \mathbb{R}$.

The infinitesimal generator of X is $\mathcal{A} = (1/2)\,(d^2/dx^2) + \mu\,(d/dx)$, and the functions of (2.5) are

$$\psi(x) = e^{\kappa x} \quad \text{and} \quad \varphi(x) = e^{\omega x}, \qquad x \in \mathbb{R},$$

respectively, where $\kappa = -\mu + \sqrt{\mu^2 + 2\beta} > 0 > \omega \triangleq -\mu - \sqrt{\mu^2+2\beta}$ are the roots of $(1/2)m^2 + \mu m - \beta = 0$. The boundaries $\pm\infty$ are natural. Observe that $\psi(-\infty) = \varphi(+\infty) = 0$ and $\psi(+\infty) = \varphi(-\infty) = +\infty$. The reward function h is continuous and

$$\ell_{-\infty} \triangleq \limsup_{x\to-\infty} \frac{h^+(x)}{\varphi(x)} = 0 \qquad \text{and} \qquad \ell_{+\infty} \triangleq \limsup_{x\to+\infty} \frac{h^+(x)}{\psi(x)} = 0.$$

The value function V is finite (cf. Proposition 5.10), and according to Proposition 5.12, $V(x) = \psi(x) W\big(G(x)\big)$, $x \in \mathbb{R}$, where $G(x) \triangleq -\varphi(x)/\psi(x) = -e^{(\omega-\kappa)x}$,

$x \in \mathbb{R}$, and $W : (-\infty, 0] \to \mathbb{R}$ is the smallest nonnegative concave majorant of

$$H(y) = \left\{ \begin{array}{ll} \frac{h}{\psi} \circ G^{-1}(y), & y < 0 \\ \ell_{+\infty}, & y = 0 \end{array} \right\} = \left\{ \begin{array}{ll} 0, & y \in (-\infty, -1] \cup \{0\} \\ \frac{(-y)^\alpha}{\omega - \kappa} \log{(-y)}, & y \in (-1, 0) \end{array} \right\},$$

where $\alpha \triangleq \frac{\kappa}{\kappa - \omega}$ $(0 < \alpha < 1)$. Note that $H(\cdot)$ is piecewise twice differentiable. In fact, $H'(y) = (-y)^{\alpha-1}\big[\alpha \log{(-y)} + 1\big]/(\kappa - \omega)$ and $H''(y) = (-y)^{\alpha-2}\big[\alpha(\alpha-1)\log{(-y)} + \alpha + (\alpha - 1)\big]/(\kappa - \omega)$ when $y \in (-1, 0)$, and they vanish on $(-\infty, -1)$. Moreover,

$$H''(y) < 0 \quad \iff \quad -e^{-(2\theta/\beta)\sqrt{\theta^2+2\beta}} \in (-1, 0).$$

and $H'(M) = 0$ gives the unique maximum $M = -e^{-1/\alpha} \in (T, 0)$ of $H(\cdot)$ (cf. Figure 6(b)).

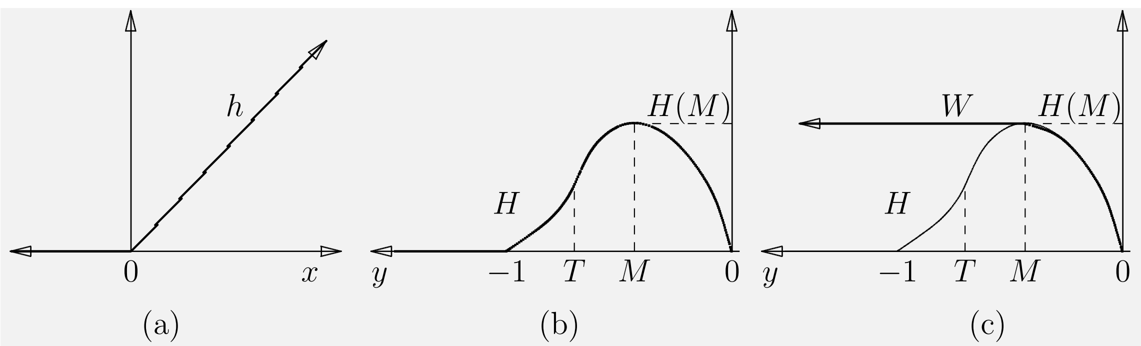

Figure 6. H. Taylor's Example

Since $H(\cdot)$ is concave on $[T, 0]$, and decreasing on $(-\infty, M]$, $M \in (T, 0)$, its smallest nonnegative concave majorant becomes

$$W(y) = \left\{ \begin{array}{ll} H(M), & y \in (-\infty, M) \\ H(y), & y \in [M, 0] \end{array} \right\}.$$

If we define $x_0 \triangleq G^{-1}(M) = 1/\alpha(\kappa - \omega) = 1/\kappa > 0$, then

$$V(x) = \psi(x) W(G(x)) = \begin{cases} e^{\kappa x - 1}/\kappa, & x < 1/\kappa, \\ x, & x \geq 1/\kappa. \end{cases}$$

Compare this with $f(\cdot)$ of Taylor [17, page 1337, Example 1] (In his notation, $a = 1/\kappa$). Finally, note that $\mathbf{C} \triangleq \{x \in \mathbb{R} : V(x) > h(x)\} = G^{-1}\big(\{y \in (-\infty, 0) : W(y) > H(y)\}\big) = G^{-1}\big((-\infty, M)\big) = (-\infty, 1/\kappa)$; and because $\ell_{-\infty} = \ell_{+\infty} = 0$, Proposition 5.13 implies

$$\tau^* \triangleq \inf\{t \geq 0 : X_t \notin \mathbf{C}\} = \inf\{t \geq 0 : X_t \geq 1/\kappa\}$$

is an optimal stopping time (although $\mathbb{P}_x\{\tau^* = +\infty\} > 0$ for $x < 1/\kappa$ if $\mu < 0$).

6.7. **An Example of P. Salminen** [16]**.** Let X be a one–dimensional Brownian motions with drift $\mu \in \mathbb{R}$. Salminen [16, page 98, Example (iii)] studies the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta\tau} h(X_\tau)], \qquad x \in \mathbb{R}$$

with the piecewise constant reward function

$$h(x) \triangleq \left\{ \begin{array}{ll} 1, & \text{if } x \leq 0 \\ 2, & \text{if } x > 0 \end{array} \right\} \equiv \left\{ \begin{array}{ll} h_1(x), & \text{if } x \leq 0 \\ h_2(x), & \text{if } x > 0 \end{array} \right\}, \quad h_1 \equiv 1,\ h_2 \equiv 2, \text{ on } \mathbb{R},$$

and discounting rate $\beta > 0$. Salminen uses Martin boundary theory (see 8) to solve the problem explicitly for $\mu = 0$.

Even though $h(\cdot)$ is not differentiable at the origin, we can use our results of Section 5 to calculate $V(\cdot)$. Note that $X_t = \mu t + B_t$, $t \geq 0$, and $X_0 = x \in \mathbb{R}$, where B is standard one–dimensional Brownian motion. Its generator is $\mathcal{A} = (1/2)\,(d^2/dx^2) + \mu\,(d/dx)$, and the functions of (2.5) are

$$\psi(x) = e^{\kappa x} \quad \text{and} \quad \varphi(x) = e^{\omega x}, \qquad x \in \mathbb{R},$$

respectively, where $\kappa \triangleq -\mu + \sqrt{\mu^2 + 2\beta} > 0 > \omega \triangleq -\mu - \sqrt{\mu^2 + 2\beta}$ are the roots of $\frac{1}{2}m^2 + \mu m - \beta = 0$. The boundaries $\pm\infty$ are natural, and $\psi(-\infty) = \varphi(+\infty) = 0$ and $\psi(+\infty) = \varphi(-\infty) = +\infty$. Note that

$$\ell_{-\infty} \triangleq \limsup_{x \to -\infty} \frac{h^+(x)}{\varphi(x)} = 0 \qquad \text{and} \qquad \ell_{+\infty} \triangleq \limsup_{x \to +\infty} \frac{h^+(x)}{\psi(x)} = 0.$$

Since $h(\cdot)$ is bounded (on every compact subset of $\mathbb{R}$), $V(\cdot)$ is finite (cf. Proposition 5.10), and $V(x) = \varphi(x) W(F(x))$, $x \in \mathbb{R}$ (cf. Proposition 5.12), where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = e^{(\kappa - \omega)x}, \qquad x \in \mathbb{R},$$

and $W : [0, \infty)$ be the smallest nonnegative concave majorant of

$$H(y) \triangleq \left\{ \begin{array}{ll} \dfrac{h}{\varphi} \circ F^{-1}(y), & y \in (0, +\infty) \\ \ell_{-\infty}, & y = 0 \end{array} \right\} = \left\{ \begin{array}{ll} H_1(y), & 0 \leq y < 1 \\ H_2(y), & y \geq 1. \end{array} \right\}$$

where $H_1(y) \triangleq y^\gamma$, $H_2(y) \triangleq 2y^\gamma$, $y \in [0, +\infty)$, and $0 < \gamma \triangleq -\omega/(\kappa - \omega) < 1$. Both $H_1(\cdot)$ and $H_2(\cdot)$ are nonnegative, strictly concave, increasing and continuously differentiable. After $y = 1$, $H(\cdot)$ switches from curve $H_1(\cdot)$ onto $H_2(\cdot)$ (Figure 7(b)).

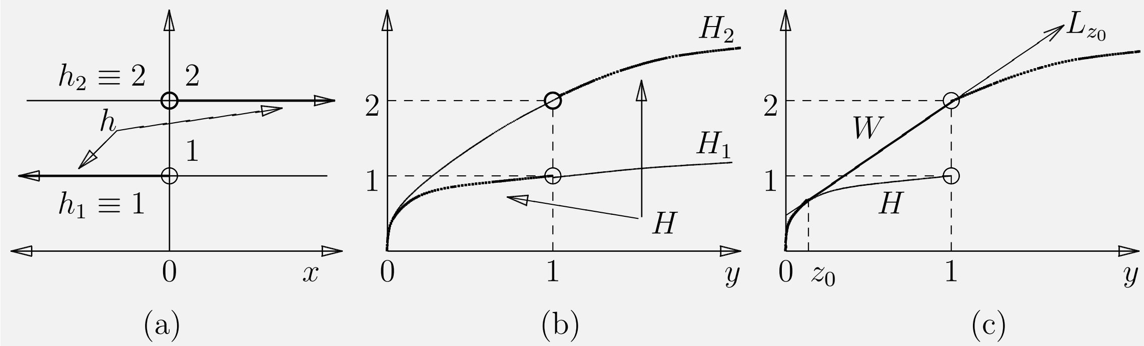

FIGURE 7. P. Salminen's Example

The strict concavity of $H(\cdot)$ on $[0,1]$, and $H'(0+) = +\infty$, imply that there exists a unique $z_0 \in (0,1)$ such that

$$H'(z_0) = \frac{H(1+) - H(z_0)}{1 - z_0} = \frac{H_2(1) - H_1(z_0)}{1 - z_0}, \tag{6.18}$$

i.e., such that the straight line $L_{z_0}(\cdot)$ tangent to $H(\cdot)$ at z_0 also passes through the point $(1, H(1+))$ (cf. Figure 7(c)). Therefore, the smallest nonnegative concave majorant of $H(\cdot)$ is

$$W(y) = \begin{cases} H(y), & y \in [0, z_0] \cup (1, +\infty), \\ L_{z_0}(y), & y \in (z_0, 1]. \end{cases}$$

If we let $x_0 \triangleq F^{-1}(z_0)$, then

$$V(x) = \varphi(x) W(F(x)) = \left\{ \begin{array}{ll} 1, & \text{if } x \leq x_0 \\ \dfrac{\big(1 - 2e^{\kappa x_0}\big)e^{\omega x} - \big(1 - 2e^{\omega x_0}\big)e^{\kappa x}}{e^{\omega x_0} - e^{\kappa x_0}}, & \text{if } x_0 < x \leq 0 \\ 2, & \text{if } x > 0 \end{array} \right\}.$$

Since $h(\cdot)$ is not continuous, we cannot use Proposition 5.13 to check if there is an optimal stopping time. However, since $\mathbf{C} \triangleq \{x \in \mathbb{R} : V(x) > h(x)\} = (x_0, 0]$, and $\mathbb{P}_0(\tau^* = 0) = 1$, we have $\mathbb{E}_0[e^{-\beta\tau^*} h(X_{\tau^*})] = h(0) = 1 < 2 = V(0)$, i.e., τ^* is not optimal. Therefore there is no optimal stopping time, either.

Salminen [16] calculates the critical value x_0 explicitly for $\mu = 0$. When we set $\mu = 0$, we get $\kappa = -\omega = \sqrt{2\beta}$, $\gamma = 1/2$, and the defining relation (6.18) of z_0 becomes

$$\frac{1}{2} z_0^{-1/2} + \frac{1}{2} z_0^{1/2} = 2 \iff z_0 - 4 z_0^{1/2} + 1 = 0,$$

after simplifications. If we let $y_0 \triangleq z_0^{1/2}$, then y_0 is the only root in $(0,1)$ of $y^2-4y+1=0$, i.e., $y_0 = 2-\sqrt{4-1} = 2-\sqrt{3}$. Therefore $z_0 = (2-\sqrt{3})^2$. Finally,

$$x_0 = F^{-1}(z_0) = \frac{1}{\kappa-\omega}\log z_0 = \frac{1}{\sqrt{2\beta}}\log(2-\sqrt{3}), \quad \text{if } \mu = 0,$$

which agrees with the calculations of Salminen [16, page 99].

6.8. **A New Optimal Stopping Problem.** Let B be one-dimensional standard Brownian motion in $[0,\infty)$ with absorption at 0. Consider

$$V(x) \triangleq \sup_{\tau\geq 0} \mathbb{E}_x[e^{-\beta\tau}(B_\tau)^p], \quad x \in [0,\infty).$$

for some $\beta > 0$ and $p > 0$. Hence our reward function $h : [0,\infty) \to \mathbb{R}$ is given as $h(x) \triangleq x^p$, which is locally bounded on $[0,+\infty)$ for any choice of $p > 0$. With $\mathcal{A} = (1/2)d^2/dx^2$, the infinitesimal generator of Brownian motion, acting on the twice–continuously differentiable functions which vanish at $\pm\infty$, the usual solutions of $\mathcal{A}u = \beta u$ are

$$\psi(x) = e^{x\sqrt{2\beta}}, \quad \text{and} \quad \varphi(x) = e^{-x\sqrt{2\beta}}, \quad x \in \mathcal{I} = \mathbb{R} \supset [0,\infty).$$

The left boundary $c = 0$ is attainable in finite time with probability one, whereas the right boundary $b = \infty$ is a natural boundary for the (stopped) process. Note that $h(\cdot)$ is continuous on $[0,\infty)$, and

$$\ell_{+\infty} \triangleq \limsup_{x\to+\infty} \frac{h^+(x)}{\psi(x)} = \lim_{x\to+\infty} \frac{h(x)}{\psi(x)} = \lim_{x\to+\infty} x^p e^{-x\sqrt{2\beta}} = 0.$$

Therefore, the value function $V(\cdot)$ is finite, and $V(x) = \psi(x)W(G(x))$, $x \in [0,\infty)$ (cf. Proposition 5.5), where $G(x) \triangleq -\varphi(x)/\psi(x) = -e^{-2x\sqrt{2\beta}}$, for every $x \in [0,\infty)$, and $W : [-1,0] \to \mathbb{R}$ is the smallest nonnegative concave majorant of

$$H(y) \triangleq \frac{h}{\psi} \circ G^{-1}(y) = \left(\frac{1}{2\sqrt{2\beta}}\right)^p \big[-\log(-y)\big]^p \cdot \sqrt{-y}, \qquad y \in [-1,0),$$

and $H(0) \triangleq \ell_{+\infty} = 0$. The function $W(\cdot)$ can be obtained analytically by cutting off the convexities of $H(\cdot)$ with straight lines (geometrically speaking, the holes on $H(\cdot)$, due to the convexity, have to be bridged across the concave hills of $H(\cdot)$, see Figure 8). Note that $H(\cdot)$ is twice continuously differentiable in $(-1,0)$; if $0 < p \leq 1$, then $H''(\cdot) \leq 0$, so $H(\cdot)$ is concave on $[-1,0]$, and $W(\cdot) = H(\cdot)$. Therefore Proposition 5.5 implies that $V(\cdot) = h(\cdot)$, and $\tau^* \equiv 0$ (i.e., stopping immediately) is optimal.

In the rest of this Subsection, we shall assume that p is strictly greater than 1. With $T \triangleq -e^{-2\sqrt{p(p-1)}}$, $H(\cdot)$ is concave on $[-1,T]$, and convex on $[T,0]$. It has unique maximum at $M \triangleq -e^{-2p} > T$, and nonnegative everywhere on $[-1, 0]$ (cf. Figure 8(a)). If $L_z(\cdot)$ is the straight line, tangent to $H(\cdot)$ at z (so-called *Smooth–Fit*

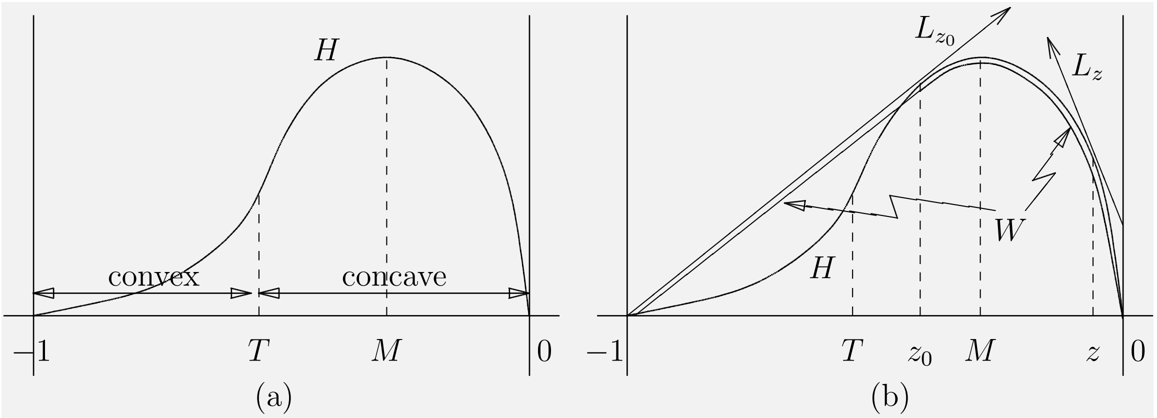

FIGURE 8. A new optimal stopping problem

in the context of variational inequalities), then there exists unique $z_0 \in [T, M)$ such that to

$$L_{z_0}(-1) = H(-1) \quad \Longleftrightarrow \quad -z_0 = e^{2p(z_0+1)/(z_0-1)},$$

and

$$W(y) = \begin{cases} L_{z_0}(y), & \text{if } y \in [-1, z_0), \\ H(y), & \text{if } y \in [z_0, 0]. \end{cases} \tag{6.19}$$

(See Figure 8(b)). Finally, $V(x) = \psi(x) W(G(x))$, for every $x \in [0, \infty)$, that is,

$$V(x) = \begin{cases} \dfrac{H(z_0)}{1+z_0} \left[e^{x\sqrt{2\beta}} - e^{-x\sqrt{2\beta}} \right], & \text{if } 0 \le x \le -\dfrac{1}{2\sqrt{2\beta}} \log(-z_0), \\ x^2, & \text{if } x > -\dfrac{1}{2\sqrt{2\beta}} \log(-z_0). \end{cases} \tag{6.20}$$

Since $\widetilde{\mathbf{C}} \triangleq \{y \in [-1.0) : W(y) > H(y)\} = (-1, z_0)$, by Remark 5.1, the optimal continuation region and optimal stopping time for our original optimal stopping problem become respectively

$$\mathbf{C} \triangleq \left(0, -\frac{1}{2\sqrt{2\beta}} \log(-z_0)\right) \quad \text{and} \quad \tau^* = \inf\left\{ t \ge 0 : B_t \ge -\frac{1}{2\sqrt{2\beta}} \log(-z_0) \right\}.$$

6.9. **Optimal Stopping Problem of Karatzas and Ocone** [11]**.** Karatzas and Ocone [11] study a special optimal stopping problem in order to solve a stochastic control problem. In this subsection, we shall take another look at the same optimal stopping problem.

Suppose that the process X is governed by the dynamics $dX_t = -\theta dt + dB_t$ for some positive constant θ, with infinitesimal generator $\mathcal{A} = (1/2)d^2/dx^2 - \theta d/dx$. Since $\pm\infty$ are natural boundaries for X, the usual solutions of $\mathcal{A}u = \beta u$, subject to the boundary conditions $\psi(-\infty) = \varphi(\infty) = 0$, become $\psi(x) = e^{\kappa x}$, $\varphi(x) = e^{\omega x}$, where $\kappa \triangleq \theta + \sqrt{\theta^2 + 2\beta}$ and $\omega \triangleq \theta - \sqrt{\theta^2 + 2\beta}$.

Now consider the stopped process, again denoted by X, which is started in $[0,\infty)$ and is absorbed when it reaches 0. Consider the optimal stopping problem

$$\inf_{\tau \in \mathcal{S}} \ \mathbb{E}_x \left[\int_0^\tau e^{-\beta t} \pi(X_t) dt + e^{-\beta\tau} g(X_\tau) \right], \quad x \in [0,\infty),$$

with $\pi(x) \triangleq x^2$ and $g(x) \triangleq \delta x^2$. If we introduce the function

(6.21)

$$R_\beta \pi(x) \triangleq \mathbb{E}_x \left[\int_0^\infty e^{-\beta t} \pi(X_t) dt \right] = \frac{1}{\beta}\, x^2 - \frac{2\theta}{\beta^2}\, x + \frac{2\theta^2 + \beta}{\beta^3} - \frac{2\theta^2 + \beta}{\beta^3} e^{\omega x}, \quad x \in [0,\infty),$$

then, the strong Markov property of X gives

$$\mathbb{E}_x \left[\int_0^\tau e^{-\beta t} \pi(X_t) dt + e^{-\beta\tau} g(X_\tau) \right] = R_\beta \pi(x) - \mathbb{E}_x [e^{-\beta\tau} (R_\beta \pi(x) - g(x))], \ x \in [0,\infty).$$

Therefore, our task is to solve the auxiliary optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x [e^{-\beta\tau} h(X_\tau)], \quad x \in [0,\infty). \tag{6.22}$$

Here, the function

$$h(x) \triangleq R_\beta \pi(x) - g(x) = \frac{1 - \delta\beta}{\beta}\, x^2 - \frac{2\theta}{\beta^2}\, x + \frac{2\theta^2 + \beta}{\beta^3} - \frac{2\theta^2 + \beta}{\beta^3}\, e^{\omega x}, \ \ x \in [0,\infty).$$

is continuous and bounded on every compact subinterval of $[0,\infty)$, and

$$\ell_\infty \triangleq \limsup_{x \to \infty} \frac{h^+(x)}{\psi(x)} = \lim_{x \to \infty} \frac{h(x)}{\psi(x)} = 0.$$

Therefore $V(\cdot)$ is finite (Proposition 5.2), and an optimal stopping time exists (Proposition 5.7). Moreover, $V(x) = \psi(x) W\big(G(x)\big)$ (Proposition 5.5), where

$$G(x) \triangleq -\frac{\varphi(x)}{\psi(x)} = -e^{(\omega - \kappa)x}, \quad x \in [0,\infty),$$

and $W(\cdot)$ is the smallest nonnegative concave majorant of

$$H(y) \triangleq \frac{h}{\psi} \circ G^{-1}(y) = (-y)^{\alpha}\Big[a\left(\log(-y)\right)^2 + b\log(-y) + c\Big] + c\,y, \quad y \in [-1, 0),$$

with $H(0) \triangleq \ell_\infty = 0$, and

$$\alpha \triangleq \frac{\kappa}{\kappa - \omega}, \quad a \triangleq \frac{1-\delta\beta}{\beta}\frac{1}{(\omega-\kappa)^2}, \quad b \triangleq -\frac{2\theta}{\beta^2}\frac{1}{(\omega-\kappa)}, \quad c \triangleq \frac{2\theta^2+\beta}{\beta^3}. \tag{6.23}$$

Observe that $0 < \alpha < 1$, $a \in \mathbb{R}$, $b \geq 0$, and $c > 0$.

We shall find $W(\cdot)$ analytically by cutting off the convexities of $H(\cdot)$. Therefore, we need to find out where $H(\cdot)$ is convex and concave. Note that $H(\cdot)$ is twice–continuously differentiable in $(-1, 0)$, and

$$H'(y) = -(-y)^{\alpha-1}\Big[\alpha a\big(\log(-y)\big)^2 + (\alpha b + 2a)\log(-y) + \alpha c + b\Big] + c, \tag{6.24}$$

$$H''(y) = (-y)^{\alpha-2}Q_1\big(\log(-y)\big), \quad y \in (-1, 0), \tag{6.25}$$

where

$$Q_1(x) \triangleq \alpha(\alpha-1)a\,x^2 + \big[\alpha(\alpha-1)b + 2a(2\alpha-1)\big]x + 2a + (2\alpha-1)b + \alpha(\alpha-1)c$$

for every $x \in \mathbb{R}$, is a second–order polynomial. Since $(-y)^{\alpha-2} > 0$, $y \in (-1, 0)$, the sign of H'' is determined by the sign of $Q_1\big(\log(-y)\big)$. Since $\log(-y) \in (-\infty, 0)$ as $y \in (-1, 0)$, we are only interested in the behavior of $Q_1(x)$ when $x \in (-\infty, 0)$. The discriminant of Q_1 becomes

$$\Delta_1 = \frac{\theta^2+\beta}{4(\theta^2+2\beta)^3\beta^2}\widetilde{Q}_1(1-\delta\beta), \tag{6.26}$$

where

$$\widetilde{Q}_1(x) \triangleq x^2 - 2x + 1 - \frac{\delta\beta^2}{\theta^2+\beta} = (x-1)^2 - \frac{\delta\beta^2}{\theta^2+\beta}, \quad x \in \mathbb{R},$$

is also a second–order polynomial, which always has two real roots,

$$\widetilde{q}_1 = 1 - \sqrt{\frac{\delta\beta^2}{\theta^2+\beta}} \quad \text{and} \quad \widetilde{q}_2 = 1 + \sqrt{\frac{\delta\beta^2}{\theta^2+\beta}}.$$

One can show that

$$\Delta_1 < 0 \quad \Longleftrightarrow \quad \delta(\theta^2+\beta) < 1.$$

Therefore, $Q_1(\cdot)$ has no real roots if $\delta(\theta^2+\beta) < 1$, has a repeated real root if $\delta(\theta^2+\beta) = 1$, and two distinct real roots if $\delta(\theta^2+\beta) > 1$. The sign of $H''(\cdot)$, and therefore the regions where $H(\cdot)$ is convex and concave, depend on the choice of the parameters δ, θ and β.

Case I. Suppose $\delta(\theta^2+\beta)<1$. Then $Q_1(\cdot)<0$, and $H''(\cdot)<0$ by (6.25). Thus $H(\cdot)$ is concave, and $W(\cdot)=H(\cdot)$. Therefore $V(\cdot)=h(\cdot)$ and the stopping time $\tau^*\equiv 0$ is optimal thanks to Propositions 5.5 and 5.7.

Suppose now $\delta(\theta^2+\beta)\geq 1$; then $Q_1(\cdot)$ has two real roots. The polynomial $Q_1(\cdot)$, and $H''(\cdot)$ by (6.25), have the same sign as $\alpha(\alpha-1)a$. Note that $\alpha(\alpha-1)$ is always negative, whereas a has the same sign as $1-\delta\beta$ thanks to (6.23).

Case II. Suppose $\delta(\theta^2+\beta)\geq 1$ and $1-\delta\beta\leq 0$. The polynomial $Q_1(\cdot)$ has two real roots $q_1\leq 0\leq q_2$; and $H(\cdot)$ is strictly concave on $[-1,-e^{q_1}]$, and strictly convex on $[-e^{q_1},0]$ $(-1<-e^{q_1}<0)$, has unique maximum at some $M\in(-1,-e^{q_1})$, and $H(M)>0$ (see Figure 9(a)). Let $L_z(y)\triangleq H(z)+H'(z)(y-z)$, $y\in[-1,0]$ be the

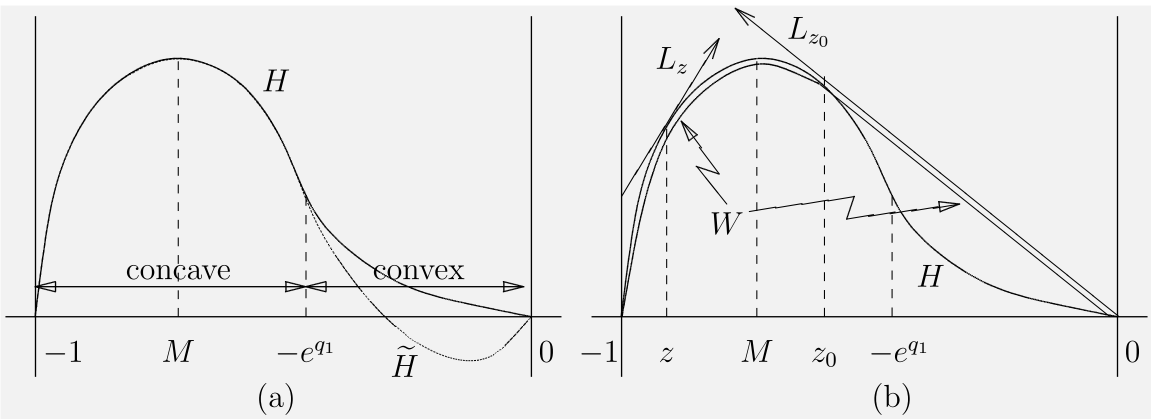

FIGURE 9. Sketches of (a) $H(\cdot)$ (may become negative in the neighborhood of zero as $\widetilde{H}$ looks like), (b) $H(\cdot)$ and W, in **Case II**.

straight line, tangent to $H(\cdot)$ at $z\in(-1,0)$; then, there exists unique $z_0\in(M,-e^{q_1}]$ such that $L_{z_0}(0)=H(0)$ (see Figure 9(b)), and the smallest nonnegative concave majorant of $H(\cdot)$ is

$$W(y)=\left\{\begin{array}{ll} H(y), & \text{if} \quad y\in[-1,z_0] \\ L_{z_0}(y), & \text{if} \quad y\in(z_0,0] \end{array}\right\}. \tag{6.27}$$

Moreover, trivial calculations show that $\log(-z_0)$ is the unique solution of

$$(1-\alpha)\big[a\,x^2+b\,x+c\big]=2ax+b,\quad x\in\big[\log(-M),q_1\big],$$

and $\widetilde{\mathbf{C}} \triangleq \{y \in [-1,0] : W(y) > H(y)\} = (z_0, 0)$ (cf. Figure 9(b)). Proposition 5.5 implies

$$V(x) = \begin{cases} h(x), & \text{if } 0 \le x \le x_0 \\ \dfrac{\varphi(x)}{\varphi(x_0)} h(x_0), & \text{if } x_0 < x < \infty \end{cases}, \tag{6.28}$$

with $x_0 \triangleq G^{-1}(z_0)$, and the optimal continuation region becomes $\mathbf{C} = G^{-1}(\widetilde{\mathbf{C}}) = G^{-1}((z_0,0)) = (x_0, \infty)$. We shall next look at the final case.

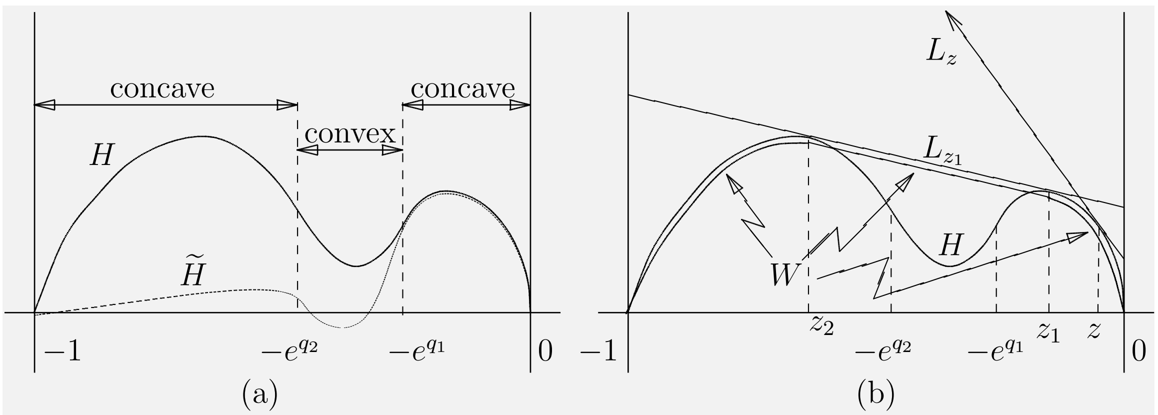

FIGURE 10. Sketches of (a) $H(\cdot)$, (b) $H(\cdot)$ and W, in **Case III**. In (a), $\widetilde{H}$ depicts another possibility where $\widetilde{H}$ takes negative values, and its global maximum is contained in $[-e^{q_1}, 0]$.

Case III. Suppose $\delta(\theta^2 + \beta) \ge 1$ and $1 - \delta\beta > 0$. The polynomial $Q_1(\cdot)$ again has two real roots $q_1 \le q_2$; and $H(\cdot)$ is convex on $(-e^{q_2}, -e^{q_1})$, and concave on $[-1,0]\backslash(-e^{q_2}, -e^{q_1})$, positive and increasing in the neighborhoods of both end–points (see Figure 10(a)). If $L_z(y) \triangleq H(z) + H'(z)(y - z)$, $y \in [-1,0]$, is the tangent line of $H(\cdot)$ at $z \in (-1,0)$, then there are unique $-1 < z_2 < z_1 < 0$, such that $L_{z_1}(\cdot)$ is tangent to $H(\cdot)$ both at z_1 and z_2, and $L_{z_1}(\cdot) \ge H(\cdot)$, on $[-1,0]$. In fact, the pair $(z, \widetilde{z}) = (z_2, z_1)$ is the unique solution of *exactly* one of the equations,

$$H'(z) = \frac{H(z) - H(\widetilde{z})}{z - \widetilde{z}} = H'(\widetilde{z}), \quad \widetilde{z} > -1, \quad \text{and} \quad H'(z) = \frac{H(z) - H(-1)}{z - (-1)}, \quad \widetilde{z} = -1,$$

for some $\widetilde{z} \in [-1, -e^{q_2}], z \in [-e^{q_1}, 0)$. Finally,

$$W(y) = \begin{cases} L_{z_1}(y), & \text{if } y \in [z_2, z_1], \\ H(y), & \text{if } y \in [-1, z_2) \cup (z_1, 0], \end{cases} \tag{6.29}$$

(Figure 10(b)). The value function $V(\cdot)$ of (6.22) follows from Proposition 5.5. Since $\widetilde{\mathbf{C}} = \{y \in [-1,0] : W(y) > H(y)\} = (z_2, z_1)$, the optimal continuation region becomes $\mathbf{C} = G^{-1}(\widetilde{\mathbf{C}}) = (G^{-1}(z_2), G^{-1}(z_1))$, and the stopping time $\tau^* \triangleq \{t \geq 0 : X_t \notin (G^{-1}(z_2), G^{-1}(z_1))\}$ is optimal.

7. Smooth–Fit Principle and Necessary Conditions for Optimal Stopping Boundaries

We shall resume in this Section our study of the properties of the value function $V(\cdot)$. For concreteness, we focus on the discounted optimal stopping problem introduced in Section 4, although all results can be carried over for the optimal stopping problems of Sections 3 and 5.

In Section 4, we started by assuming that $h(\cdot)$ is bounded, and showed that $V(\cdot)/\varphi(\cdot)$ is the smallest nonnegative F–concave majorant of $h(\cdot)/\varphi(\cdot)$ on $[c,d]$ (cf. Proposition 4.2); the continuity of $V(\cdot)$ in (c,d) then followed from concavity. The F–concavity property of $V(\cdot)/\varphi(\cdot)$ has further implications. From Proposition 2.6(iii), we know that $D_F^{\pm}(V/\varphi)$ exist and are nondecreasing in (c,d). Furthermore,[3]

$$\frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) \geq \frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x), \quad x \in (c,d). \tag{7.1}$$

Proposition 2.6(iii) implies that equality holds in (7.1) everywhere in (c,d), except possibly on a subset N which is at most countable, i.e.,

$$\frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) \equiv \frac{d}{dF}\left(\frac{V}{\varphi}\right)(x), \quad x \in (c,d)\backslash N.$$

Hence $V(\cdot)/\varphi(\cdot)$ is essentially F–differentiable in (c,d). Let

$$\mathbf{\Gamma} \triangleq \{x \in [c,d] : V(x) = h(x)\} \quad \text{and} \quad \mathbf{C} \triangleq [c,d]\backslash \mathbf{\Gamma} = \{x \in [c,d] : V(x) > h(x)\}.$$

When the F–concavity of $V(\cdot)/\varphi(\cdot)$ is combined with the fact that $V(\cdot)$ majorizes $h(\cdot)$ on $[c,d]$, we obtain the key result of Proposition 7.1, which leads, in turn, to the celebrated Smooth–Fit principle.

Proposition 7.1. *At every $x \in \mathbf{\Gamma} \cap (c,d)$, where $D_F^{\pm}\big(h/\varphi\big)(x)$ exist, we have*

$$\frac{d^-}{dF}\left(\frac{h}{\varphi}\right)(x) \geq \frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) \geq \frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x) \geq \frac{d^+}{dF}\left(\frac{h}{\varphi}\right)(x).$$

[3] The fact that the left–derivative of the value function $V(\cdot)$ is always greater than or equal to the right–derivative of $V(\cdot)$ was pointed by Salminen [16, page 86].

Proof. The second inequality is the same as (7.1). For the rest, first remember that $V(\cdot) = h(\cdot)$ on $\mathbf{\Gamma}$. Since $V(\cdot)$ majorizes $h(\cdot)$ on $[c,d]$, and $F(\cdot)$ is strictly increasing, this leads to

$$\frac{\frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)}}{F(y)-F(x)} \geq \frac{\frac{V(y)}{\varphi(y)} - \frac{V(x)}{\varphi(x)}}{F(y)-F(x)} \quad \text{and} \quad \frac{\frac{V(z)}{\varphi(z)} - \frac{V(x)}{\varphi(x)}}{F(z)-F(x)} \geq \frac{\frac{h(z)}{\varphi(z)} - \frac{h(x)}{\varphi(x)}}{F(z)-F(x)}, \tag{7.2}$$

for every $x \in \mathbf{\Gamma}$, $y < x < z$. Suppose $x \in \mathbf{\Gamma} \cap (c,d)$, and $D_F^{\pm}\big(h/\varphi\big)(x)$ exist. As we summarized before stating Proposition 7.1, we know that $D_F^{\pm}\big(V/\varphi\big)(x)$ always exist in (c,d). Therefore, the limits of both sides of the inequalities in (7.2), as $y \uparrow x$ and $z \downarrow x$ respectively, exist, and give $D_F^-\left(h/\varphi\right)(x) \geq D_F^-\left(V/\varphi\right)(x)$, and $D_F^+\left(V/\varphi\right)(x) \geq D_F^+\left(h/\varphi\right)(x)$, respectively. □

Corollary 7.1 (Smooth–Fit Principle)**.** *At every $x \in \mathbf{\Gamma} \cap (c,d)$ where $h(\cdot)/\varphi(\cdot)$ is F–differentiable, $V(\cdot)/\varphi(\cdot)$ is also F–differentiable, and touches $h(\cdot)/\varphi(\cdot)$ at x smoothly, in the sense that the F–derivatives of both functions also agree at x:*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(x).$$

Corollary 7.1 raises the question when we should expect $V(\cdot)/\varphi(\cdot)$ to be F–differentiable in (c,d). If $h(\cdot)/\varphi(\cdot)$ is F–differentiable in (c,d), then it is immediate from Corollary 7.1 that $V(\cdot)/\varphi(\cdot)$ is F–differentiable in $\mathbf{\Gamma} \cap (c,d)$. However, we know little about the behavior of $V(\cdot)/\varphi(\cdot)$ on $\mathbf{C} = [c,d] \backslash \mathbf{\Gamma}$ if $h(\cdot)$ is only bounded. If, however, $h(\cdot)$ is continuous on $[c,d]$, then $V(\cdot)$ is also continuous on $[c,d]$ (cf. Lemma 4.2), and now $\mathbf{C}$ is an open subset of $[c,d]$. Therefore, it is the union of a countable family $(J_\alpha)_{\alpha \in \Lambda}$ of disjoint open (relative to $[c,d]$) subintervals of $[c,d]$. By Lemma 4.3,

$$\frac{V(x)}{\varphi(x)} = \frac{\mathbb{E}_x[e^{-\beta\tau^*}h(X_{\tau^*})]}{\varphi(x)} = \frac{V(l_\alpha)}{\varphi(l_\alpha)} \cdot \frac{F(r_\alpha)-F(x)}{F(r_\alpha)-F(l_\alpha)} + \frac{V(r_\alpha)}{\varphi(r_\alpha)} \cdot \frac{F(x)-F(l_\alpha)}{F(r_\alpha)-F(l_\alpha)}, \quad x \in J_\alpha, \tag{7.3}$$

where l_α and r_α are the left– and right–boundary of J_α, $\alpha \in \Lambda$, respectively. Observe that $V(\cdot)/\varphi(\cdot)$ coincides with an F–linear function on every J_α, i.e., it is F–differentiable in $J_\alpha \cap (c,d)$ for every $\alpha \in \Lambda$. By taking the F–derivative of (7.3), we find that

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{1}{F(r_\alpha)-F(l_\alpha)}\left[\frac{V(r_\alpha)}{\varphi(r_\alpha)} - \frac{V(l_\alpha)}{\varphi(l_\alpha)}\right], \quad x \in J_\alpha \cap (c,d) \tag{7.4}$$

is constant, i.e., is itself F–differentiable in $J_\alpha \cap (c,d)$. Since $\mathbf{C}$ is the union of disjoint J_α, $\alpha \in \Lambda$, this implies that $V(\cdot)/\varphi(\cdot)$ is twice continuously F–differentiable in $\mathbf{C} \cap (c,d)$. We are ready to prove the following result.

Proposition 7.2. *Suppose $h(\cdot)$ is continuous on $[c,d]$. Then $V(\cdot)$ is continuous on $[c,d]$, and $V(\cdot)/\varphi(\cdot)$ is twice continuously F–differentiable in $\mathbf{C}\cap(c,d)$. Furthermore,*

(i) *if $h(\cdot)/\varphi(\cdot)$ is F–differentiable in (c,d), then $V(\cdot)/\varphi(\cdot)$ is* **continuously**[4] *F–differentiable in (c,d), and*

(ii) *if $h(\cdot)/\varphi(\cdot)$ is twice (continuously) F–differentiable in (c,d), then $V(\cdot)/\varphi(\cdot)$ is twice (continuously) F–differentiable in $(c,d)\backslash\partial\mathbf{C}$,*

where $\partial\mathbf{C}$ is the boundary of $\mathbf{C}$ relative to $\mathbb{R}$ or $[c,d]$.

Proof. Since $h(\cdot)$ and F are continuous, $V(\cdot)$ is continuous by Lemma 4.2. We also proved above that $V(\cdot)/\varphi(\cdot)$ is F–differentiable in $\mathbf{C}\cap(c,d)$ (this is always true even if $h(\cdot)/\varphi(\cdot)$ were not F–differentiable).

(i) If $h(\cdot)/\varphi(\cdot)$ is F–differentiable in (c,d), then the F–differentiability of $V(\cdot)/\varphi(\cdot)$ in $(c,d)\backslash\mathbf{C} = (c,d)\cap\mathbf{\Gamma}$ follows from Corollary 7.1. Therefore $V(\cdot)/\varphi(\cdot)$ is F–differentiable in $(c,d) = [(c,d)\backslash\mathbf{C}]\cup\mathbf{C}$ by the discussion above. However, $V(\cdot)/\varphi(\cdot)$ is also F–concave on $[c,d]$, and F is continuous on $[c,d]$. Therefore $D_F\left(V/\varphi\right)(\cdot)$ is continuous on (c,d).

(ii) We only need prove that $V(\cdot)/\varphi(\cdot)$ is twice (continuously) F–differentiable in $(c,d)\backslash\overline{\mathbf{C}}$ where $\overline{\mathbf{C}}$ is the closure of $\mathbf{C}$ relative to $[c,d]$. However $(c,d)\backslash\overline{\mathbf{C}}$ is an open set (relative to $\mathbb{R}$) contained in $\mathbf{\Gamma}$ where $V(\cdot)$ and $h(\cdot)$ coincide. Because we assume $h(\cdot)/\varphi(\cdot)$ is twice (continuously) F–differentiable, the conclusion follows immediately. □

Proposition 7.3 (Necessary conditions for the boundaries of the optimal continuation region)**.** *Suppose $h(\cdot)$ is continuous on $[c,d]$. Suppose $l,r\in\mathbf{\Gamma}\cap(c,d)$, and $h(\cdot)/\varphi(\cdot)$ has F–derivatives at l and r. Then $D_F\big(V/\varphi\big)(\cdot)$ exists at l and r. Moreover, we have the following cases:*

(i) *If $(l,r)\subseteq\mathbf{C}$, then*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(l) = \frac{\frac{h(r)}{\varphi(r)}-\frac{h(l)}{\varphi(l)}}{F(r)-F(l)} = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(r) = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r),$$

and,

$$\frac{V(x)}{\varphi(x)} = \frac{h(\theta)}{\varphi(\theta)} + \big[F(x)-F(\theta)\big]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(\theta), \quad x\in[l,r],\ \theta = l,r.$$

[4] Note that this is always true no matter whether $D_F(h/\varphi)$ is continuous or not. As the proof indicates, this is as a result of F–concavity of $V(\cdot)/\varphi(\cdot)$ and continuity of F on $[c,d]$.

(ii) *If* $[c,r)\subseteq \mathbf{C}$, *then*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(r)=\frac{d}{dF}\left(\frac{V}{\varphi}\right)(r)=\frac{1}{F(r)-F(c)}\cdot\frac{h(r)}{\varphi(r)},$$

and,

$$\frac{V(x)}{\varphi(x)}=\frac{h(r)}{\varphi(r)}+\big[F(x)-F(r)\big]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(r)=\big[F(x)-F(c)\big]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(r),\quad x\in[c,r).$$

(iii) *If* $(l,d]\subseteq \mathbf{C}$, *then*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l)=\frac{d}{dF}\left(\frac{V}{\varphi}\right)(l)=-\frac{1}{F(d)-F(l)}\cdot\frac{h(l)}{\varphi(l)},$$

and,

$$\frac{V(x)}{\varphi(x)}=\frac{h(l)}{\varphi(l)}+\big[F(x)-F(l)\big]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l)=\big[F(x)-F(d)\big]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l),\quad x\in(l,d].$$

Proof. The existence of $D_F\big(V/\varphi\big)$, and its equality with $D_F\big(h/\varphi\big)$ at l and r, follow from Corollary 7.1. Therefore, the first and last equality in (i), and the first equalities in (ii) and (iii) are clear.

Note that the intervals (l,r), $[c,r)$ and $(l,b]$ are all three possible forms that J_α, $\alpha\in\Lambda$ can take. Let l_α and r_α denote the left– and right–boundaries of intervals, respectively. Then (7.4) is true for all three cases.

In (i), both $l_\alpha=l$ and $r_\alpha=r$ are in $\mathbf{\Gamma}$. Therefore, $V(l)=h(l)$ and $V(r)=h(r)$, and (7.4) implies

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x)=\frac{1}{F(r)-F(l)}\left[\frac{h(r)}{\varphi(r)}-\frac{h(l)}{\varphi(l)}\right],\quad x\in(l,r). \tag{7.5}$$

Since $V(\cdot)/\varphi(\cdot)$ is F–concave on $[c,d]\supset[l,r]$, and F is continuous on $[c,d]$, Proposition 2.6(iii) implies that $D_F^+\left(V/\varphi\right)$ and $D_F^-\left(V/\varphi\right)$ are right– and left–continuous in (c,d). Because $V(\cdot)/\varphi(\cdot)$ is F–differentiable on $[l,r]$, $D_F^\pm\left(V/\varphi\right)$ and $D_F\left(V/\varphi\right)$ coincide on $[l,r]$. Therefore $D_F\left(V/\varphi\right)$ is continuous on $[l,r]$, and second and third equalities in (i) immediately follow from (7.5). In a more direct way,

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(l)=\frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(l)=\lim_{x\downarrow l}\frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x)=\lim_{x\downarrow l}\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x)=\frac{\frac{h(r)}{\varphi(r)}-\frac{h(l)}{\varphi(l)}}{F(r)-F(l)}.$$

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(r)=\frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(r)=\lim_{x\downarrow l}\frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x)=\lim_{x\downarrow l}\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x)=\frac{\frac{h(r)}{\varphi(r)}-\frac{h(l)}{\varphi(l)}}{F(r)-F(l)}.$$

Same equalities could have also been proved by direct calculation using (7.3).

The proofs of the second equalities in (ii) and (iii) are similar, once we note that $V(c) = 0$ if $c \in \mathbf{C}$, and $V(d) = 0$ if $d \in \mathbf{C}$. Finally, the expressions for $V(\cdot)/\varphi(\cdot)$ follow from (7.3) by direct calculations; simply note that $V(\cdot)/\varphi(\cdot)$ is an F–linear function passing through $\big(l_\alpha, (V/\varphi)(l_\alpha)\big)$ and $\big(r_\alpha, (V/\varphi)(r_\alpha)\big)$. □

We shall verify that our necessary conditions agree with those of Salminen [16, Theorem 4.7]. To do this, we first remember his

Definition 7.1 (Salminen [16], page 95)**.** *A point $x^* \in \mathbf{\Gamma}$ is called a* **left boundary** *of $\mathbf{\Gamma}$ if for $\varepsilon > 0$ small enough $(x^*, x^* + \varepsilon) \subseteq \mathbf{C}$ and $(x^* - \varepsilon, x^*] \subseteq \mathbf{\Gamma}$. A point $y^* \in \mathbf{\Gamma}$ is called a* **right boundary** *of $\mathbf{\Gamma}$ if for $\varepsilon > 0$ small enough $(y^* - \varepsilon, y^*) \subseteq \mathbf{C}$ and $[y^*, y^* + \varepsilon) \subseteq \mathbf{\Gamma}$ (cf. Figure 11 for illustration).*

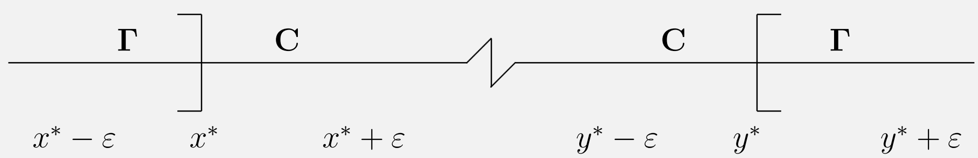

FIGURE 11. x^* is a left- and y^* is a right-boundary point of $\mathbf{\Gamma}$.

We shall also remind the definitions of the key functions $G_b(\cdot)$ and $G_a(\cdot)$ of Salminen's conclusion. At every $x \in (c, d)$ where $h(\cdot)$ is S–differentiable, let

$$G_b(x) \triangleq \varphi(x)\frac{dh}{dS}(x) - h(x)\frac{d\varphi}{dS}(x) \quad \text{and} \quad G_a(x) \triangleq h(x)\frac{d\psi}{dS} - \psi(x)\frac{dh}{dS}(x). \tag{7.6}$$

Proposition 7.4. *Suppose $h(\cdot)$ is continuous on $[c, d]$. If $h(\cdot)$, $\psi(\cdot)$ and $\varphi(\cdot)$ are S–differentiable at some $x \in (c, d)$, then $h(\cdot)/\varphi(\cdot)$ and $h(\cdot)/\psi(\cdot)$ are F– and G–differentiable at x, respectively. Moreover,*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x) = \frac{G_b(x)}{W(\psi, \varphi)} \quad \text{and} \quad \frac{d}{dG}\left(\frac{h}{\psi}\right)(x) = -\frac{G_a(x)}{W(\psi, \varphi)}, \tag{7.7}$$

where $G_b(x)$ and $G_a(x)$ are defined as in (7.6), and the Wronskian $W(\psi, \varphi) \triangleq \varphi(\cdot)\frac{d\psi}{dS}(\cdot) - \psi(\cdot)\frac{d\varphi}{dS}(\cdot)$ is constant and positive (cf. Section 2).

Proof. Since $h(\cdot)$, $\psi(\cdot)$ and $\varphi(\cdot)$ are S–differentiable at x, $h(\cdot)/\varphi(\cdot)$ and F are S–differentiable at x. Therefore, $D_F\big(h/\varphi\big)$ exist at x, and equals

$$\begin{aligned}\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x) &= \frac{\frac{d}{dS}\left(\frac{h}{\varphi}\right)}{\frac{dF}{dS}}(x) = \frac{D_S h\cdot\varphi - h\cdot D_S\varphi}{D_S\psi\cdot\varphi-\psi\cdot D_S\varphi}(x)\\ &= \frac{1}{W(\psi,\varphi)}\left[\varphi(x)\,\frac{dh}{dS}(x) - h(x)\,\frac{d\varphi}{dS}(x)\right] = \frac{G_b(x)}{W(\psi,\varphi)},\end{aligned} \tag{7.8}$$

where $D_S \equiv \frac{d}{dS}$. Noting the symmetry in (φ, F) versus (ψ, G), we can repeat all arguments by replacing (φ,ψ) with $(\psi,-\varphi)$. Therefore it can be similarly shown that $D_G\big(h/\psi\big)(x)$ exists and $D_G\big(h/\psi\big)(x) = -G_a(x)/W(\psi,\varphi)$ (note that $W(-\varphi,\psi) = W(\psi,\varphi)$). □

Corollary 7.2 (Salminen [16], Theorem 4.7)**.** *Let $h(\cdot)$ be continuous on $[c,d]$. Suppose l and r are left– and right–boundary points of $\mathbf{\Gamma}$, respectively, such that $(l,r)\subseteq\mathbf{C}$. Assume that $h(\cdot)$, $\psi(\cdot)$ and $\varphi(\cdot)$ are S(scale function)–differentiable on the set $A \triangleq (l-\varepsilon, l]\cup[r, r+\varepsilon)$ for some $\varepsilon>0$ such that $A\subseteq\mathbf{\Gamma}$. Then on A, the functions G_b and G_a of (7.6) are non–increasing and non–decreasing, respectively, and*

$$G_b(l) = G_b(r), \quad G_a(l) = G_a(r).$$

Proof. Proposition 7.4 implies that $D_F\big(h/\varphi\big)$ and $D_G\big(h/\psi\big)$ exist on A. Since $l,r\in\mathbf{\Gamma}$ and $(l,r)\subseteq\mathbf{C}$, Proposition 7.3(i) and (7.7) imply

$$\frac{G_b(l)}{W(\psi,\varphi)} = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) = \frac{G_b(r)}{W(\psi,\varphi)},$$

i.e., $G_b(l) = G_b(r)$ (Remember also that the Wronskian $W(\psi,\varphi)\triangleq\frac{d\psi}{dS}\varphi - \psi\frac{d\varphi}{dS}$ of $\psi(\cdot)$ and $\varphi(\cdot)$ is a positive constant; see Section 2). By symmetry in the pairs (φ,F) and (ψ,G), we have similarly $G_a(l) = G_a(r)$.

On the other hand, observe that $D_F\big(V/\varphi\big)$ and $D_G\big(V/\psi\big)$ also exist and, are equal to $D_F\big(h/\varphi\big)$ and $D_G\big(h/\psi\big)$ on A, respectively, by Corollary 7.1. Therefore

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{G_b(x)}{W(\psi,\varphi)} \quad\text{and}\quad \frac{d}{dG}\left(\frac{V}{\psi}\right)(x) = -\frac{G_a(x)}{W(\psi,\varphi)}, \quad x\in A, \tag{7.9}$$

by Proposition 7.7. Because $V(\cdot)/\varphi(\cdot)$ is F–concave, and $V(\cdot)/\psi(\cdot)$ is G–concave, Proposition 2.6(i) implies that both $D_F\big(V/\varphi\big)$ and $D_G\big(V/\psi\big)$ are non–increasing on A. Therefore (7.9) implies that G_b is non–increasing, and G_a is non–decreasing on A. □

8. Concluding Remarks: Martin Boundary Theory and Optimal Stopping for Markov Processes in General

We shall conclude by pointing out the importance of Martin boundary theory (cf. Dynkin [6, 7]) in the study of optimal stopping problems for Markov processes. This indicates that every excessive function of a Markov process can be represented as the integral of minimal excessive functions with respect to a unique representing measure. If the process X is a regular one–dimensional diffusion with state space $\mathcal{I}$, whose end–points are a and b, then Salminen [16, Theorem 2.7] shows that the minimal β–excessive functions are

$$k_a(\cdot) \triangleq \varphi(\cdot), \quad k_b(\cdot) \triangleq \psi(\cdot), \quad k_y(\cdot) \triangleq \frac{\psi(\cdot)}{\psi(y)} \wedge \frac{\varphi(\cdot)}{\varphi(y)}, \quad \forall\, y \in (a,b).$$

Then, according to Martin boundary theory, every β–excessive function $h(\cdot)$ can be represented as

$$h(x) = \int_{[a,b]} k_y(x)\nu_h(dy), \qquad x \in \mathcal{I}, \tag{8.1}$$

where ν_h is a finite measure on $[a,b]$, uniquely determined by $h(\cdot)$. Now, observe that $k_y(\cdot)/\varphi(\cdot)$ is F–concave for every $y \in [a,b]$. Therefore, Proposition 4.1 and its counterparts in Section 5, can also be seen as consequences of the representation (8.1). The functions $\psi(\cdot)$, $\varphi(\cdot)$ are harmonic functions of the process X killed at an exponentially distributed independent random time, and are associated with points in the Martin boundary of the killed process. The Martin boundary has been studied widely in the literature for arbitrary Markov processes, and seems the right tool to use, if one tries to extend the results of this paper to optimal stopping of general Markov processes. Such an extension is currently being investigated by the authors.

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