INDEX POLICIES FOR DISCOUNTED BANDIT PROBLEMS WITH AVAILABILITY CONSTRAINTS

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Abstract. In the classical bandit problem, the arms of a slot machine are always available. This paper studies the case where the arms are not always available. We first consider the problem where the arms are intermittently available with some time-dependent probabilities. We prove the non-existence of an optimal index policy and propose the so-called Whittle index policy after reformulating the problem as a restless bandit. The index strikes the balance between exploration and exploitation: it converges to the Gittins index as the probability of availability approaches to one and to the immediate one-time reward as it approaches to zero. We then consider the problem where the arms may break down and repair is an option at some cost, and we derive the corresponding Whittle index policy. We show that both problems are indexable and that the proposed index policies cannot be dominated uniformly by any other index policy over the entire class of bandit problems considered here. We illustrate how to evaluate one of the indices on a numerical example in which rewards are Bernoulli random variables with unknown success probabilities.

1. Introduction

The multi-armed bandit problem considers the trade-off between exploration and exploitation. It deals with the situations in which one needs to decide between actions that maximize immediate reward and actions acquiring information that may help increase one's total reward in the future. In the classical multi-armed bandit problem, the decision maker has to choose at each stage one arm of an N-armed slot machine to play in order to maximize his expected total discounted reward over infinite time horizon. The reward obtained from an arm depends on the state of a stochastic process that changes only when the arm is played.

Gittins and Jones [5] showed that each arm is associated with an index that is a function of the state of the arm, and that the expected total discounted reward over infinite-horizon is maximized if every time an arm with the largest index is played. We call an arm *active* if it is played and *passive* otherwise. The proof (see, for example, Whittle [22] and Tsitsiklis [19]) relies on the condition that only active arms change their states. Due to this limitation, the

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range of problems where optimal index policies are guaranteed to exist is small. Nevertheless, the Gittins index policy is important because it considerably reduces the dimension of the optimization problem by splitting it into N independent subproblems. Moreover, since at each stage only one arm changes its state, at most one index has to be re-evaluated. For this reason, many authors have generalized the classical multi-armed bandit problem and studied performance of Gittins-like index policies for them.

In this paper, we study bandit problems where passive *and* active arms may become unavailable temporarily or permanently. Therefore, these are not classical multi-armed bandit problems, and the Gittins index policies are not optimal any more.

For example in clinical trials, one chooses sequentially between alternative medicines to prescribe to a number of patients. The arms correspond to medicines, and the state of each arm corresponds to the state of knowledge about the corresponding medicine's efficacy. There are, however, situations where some medicine cannot be applied to certain patients. For example, some medicines may go out of stock, or some patients are allergic to certain ingredients of some medicines.

The bandit problems are also common in economics. In the face of the trade-off between exploration and exploitation, rational decision makers are assumed to act optimally using Gittins index policy. This framework has been used to explain, for example, insufficient learning (Rothschild [18], Banks and Sundaram [1], and Brezzi and Lai [4]), matching and job search (Jovanovic [10] and Miller [14]) and mechanism design (Bergemann and Valimaki [3]). However, decision makers do not act in the same way when alternatives may become unavailable in the future. Intuitively, the more pessimistic a decision maker is about the future availability of the alternatives, the more focused he gets to immediate payoffs. Therefore, it is questionable to expect that decision makers use the Gittins index policy in the situations where the alternatives occasionally become unavailable.

In a variation of the above-mentioned problem, we assume that arms may break down, and that the decision maker has the option to fix them. For example, if an energy company loses the access to oil due to an unexpected international conflict, will it be worth the effort to get another access to oil or will it be better to rely on other resources available to the company such as natural gas or coal? Bandit problems with switching costs (see, for example, Jun [11]) is a special case; arms break down immediately if they are not engaged, and if a broken arm is engaged, then the switching cost is incurred to repair. According to Bank and Sundaram [2], it is difficult to imagine an economic problem where the agent can switch costlessly between alternatives. They also showed that an optimal index policy does not exist in the presence of switching costs.

We generalize the classical multi-armed bandit problem as follows. There are N arms, and each arm is available with some state-dependent probability. At each decision stage, the decision maker chooses M arms to play simultaneously and collects rewards from each arm. The reward from arm n depends on a stochastic process $X_n = (X_n(t))_{t\geq 0}$, whose state changes only when the arm is played. The process X_n may represent, for example, the state of the knowledge about the reward obtainable from arm n.

At every iteration, only a subset of the arms are available. We denote by $Y_n(t)$ the availability of arm n at time t; it is one if the arm is available at time t, and zero otherwise. Unlike X_n, the stochastic process Y_n changes even when the arm is not played. The objective is to find a policy that chooses arms so as to maximize the expected total discounted reward collected over infinite horizon. We study the following two problems:

Problem 1. *Each arm is intermittently available. Its availability at time t is unobservable before time t. The conditional probability that an arm is available at time $t+1$ given*

(i) *the state $X(t)$ and its availability $Y(t)$ of the arm, and*
(ii) *whether or not the arm is played at time t*

is known at time t. An arm cannot be played when it is unavailable.

This problem will not be well-defined unless there are at least M available arms to play at each stage. We can, however, let the decision maker pull less than M arms at a time by introducing sufficient number of arms that are always available and always give zero reward.

Problem 2. *Arms are subject to failure, and the decision maker has the option to repair a broken arm. Whether or not an arm is played at time t, it may break down and may not be available at time $t+1$ with some probability that depends on*

(i) *the state $X(t)$ of the arm at time, and*
(ii) *whether or not the arm is played at time t.*

If an arm is broken, then the decision maker has the option to repair it at some cost (or negative reward) that depends on $X(t)$. Repairing an arm is equivalent to playing the arm when it is broken. If an arm is repaired at time t, then it will be available at time $t+1$ with some conditional probability that depends only on the state $X(t)$ of the arm at time t. On the other hand, if it is not repaired, then the arm remains broken at time $t+1$.

We show that an optimal index policy does not exist for either problem. Nevertheless, there exists a near-optimal index policy that cannot be dominated uniformly by any other index policy over all instances of either problem. The index we propose is based on the Whittle index proposed for the restless bandit problems. We evaluate the performance of the index policy for each problem both analytically and numerically.

The restless bandit problem was introduced by Whittle [23], and it is a generalization of the classical bandit problem in three directions: (i) the states of passive arms may change, (ii) rewards can be collected from passive arms, and (iii) $M \geq 1$ arms can be played simultaneously. Therefore, Problems 1 and 2 fall in the class of restless bandit problems. These problems are computationally intractable; Papadimitriou and Tsitsiklis [16] proved that they are PSPACE-hard. As in a typical restless bandit problem, we assume that rewards can be collected from passive arms and that more than one arm can be pulled simultaneously.

Whittle [23] introduced the so-called Whittle index to maximize long-term average reward and characterized the index as a Lagrange multiplier for a relaxed conservation constraint, which ensures that on average M arms are played at each stage. The Whittle index policy makes sense if the problem is *indexable*. Weber and Weiss [20, 21] proved that under indexability, the Whittle index policy is asymptotically optimal as M and N tend to infinity while M/N is constant. The verification of indexability is hard in general. Whittle [23] gave an example of a unindexable problem. However, indexability can be verified, and Whittle index policy can be developed analytically, for example, for the dual-speed restless bandit problem [7] and the problem in [8] with improving active arms and deteriorating passive arms.

Glazebrook et al. [6] considered a problem in which passive arms are subject to permanent failure. They modeled it as a restless bandit, showed its indexability, and developed the Whittle index policy. Problems 1 and 2 are the generalizations of their problem in that a broken arm is allowed to get back to the system, and that both passive *and* active arms may break down. We prove that Problems 1 and 2 are indexable and derive the Whittle indices. Glazebrook et al.'s [6] and Gittins' indices turn out to be the special cases of those indices.

We also evaluate the Whittle index numerically. Like the Gittins index, the Whittle indices for Problems 1 and 2 are also the solutions to suitable optimal stopping problems. We generalize Katehakis and Veinott's [12] restart-in formulation of Gittins index to Problem 1's Whittle index. Problem 2's Whittle index turns out to be similar to the Gittins index, and we use the original restart-in problem to calculate the index for Problem 2.

In Section 2, we start by modeling Problems 1 and 2 as restless bandits. In Section 3, we review Whittle index and indexability. In Sections 4 and 5, we verify the indexability of Problems 1 and 2 and develop corresponding Whittle indices; we also prove the nonexistence of an optimal index policy. A generalization of the restart-in problem that is used to calculate Problem 1's Whittle index is discussed in Section 6. We then consider a numerical example in which the reward process is a Bernoulli process with unknown success probability, and we evaluate the index policy. The example is introduced in Section 7, and the results for Problems 1 and 2 are given in Sections 8 and 9, respectively. Section 10 contains the concluding remarks.

2. Model

We first model a general restless bandit problem that subsumes Problems 1 and 2, and then specialize to the latter ones in Sections 4 and 5, respectively.

Using the stochastic processes X and Y defined in the previous section, the state of arm n at time t can be denoted by $S_n(t) = (X_n(t), Y_n(t))$. We denote by $S(t) = (S_1(t), S_2(t), \ldots, S_N(t))$ the state at time t of the system consisting of N arms.

Suppose that X_n takes values in $\mathcal{X}_n, n = 1, \ldots, N$, and let $\mathcal{S}_n = \mathcal{X}_n \times \{0, 1\}$. The process $(X_n(t), Y_n(t))$ is a controlled time-homogeneous Markov process with the control

$$a_n(t) = \begin{cases} 1, & \text{if arm } n \text{ is played at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a(t) = (a_1(t), a_2(t), \ldots, a_N(t))$ denote the control action at time $t \geq 0$.

For every $1 \leq n \leq N$, the process $(X_n(t))_{t \in \geq 0}$ evolves according to a one-step transition probability matrix $p^{(n)} = (p^{(n)}_{xx'})_{x,x' \in \mathcal{X}_n}$, if arm n is available and is played, and does not change otherwise; that is, for every $x, x' \in \mathcal{X}_n$,

$$\mathbb{P}\left\{X_n(t+1) = x' | X_n(t) = x, Y_n(t) = y, a_n(t) = a\right\} = \begin{cases} p^{(n)}_{xx'}, & \text{if } y = a = 1, \\ \delta_{xx'}, & \text{if } y = 0 \text{ or } a = 0, \end{cases}$$

where $\delta_{xx'}$ equals one if $x = x'$ and zero otherwise. Hence, even if arm n is active, the process X_n does not change if the arm is unavailable. In Problem 2, activating an unavailable arm is equivalent to repairing it. In that case, the process X_n does not change; namely, repairing an arm only changes its availability. In Problem 1, activating an unavailable arm is not allowed.

We denote the conditional probability that arm n is available at time $t+1$, given $X_n(t)$, $Y_n(t)$, and $a_n(t)$ by

$$\theta^a_n(x, y) \triangleq \mathbb{P}\left\{Y_n(t+1) = 1 | X_n(t) = x, Y_n(t) = y, a_n(t) = a\right\}, \tag{1}$$

for every $(x, y) \in \mathcal{S}_n$, $a \in \{0, 1\}$, $t \geq 0$, and $1 \leq n \leq N$. In other words, the random variable $Y_n(t+1)$ has conditionally Bernoulli distribution with success probability $\theta^{a(t)}_n(X_n(t), Y_n(t))$ given $X_n(t)$ and $Y_n(t)$. For every $x \in \mathcal{X}_n$ and $a, y \in \{0, 1\}$, let

$$R^a_n(x, y) \triangleq \text{expected reward collected from arm } n \text{ when } X_n(t) = x, Y_n(t) = y, a_n(t) = a,$$

and as in the classical bandit problem, we assume that $R^a_n(x, y)$ is bounded uniformly over $x \in \mathcal{X}$ and $y \in \{0, 1\}$. Let $0 < \gamma < 1$ be a given discount rate. Then the expected discounted immediate reward at time t equals $\mathbb{E}\left[\gamma^t \sum_{n=1}^N R^{a_n(t)}_n(X_n(t), Y_n(t))\right]$. If $a_i(t) = 1$ and $Y_i(t) = 1$, then $X_i(t)$ changes to $X_i(t+1)$ according to $p^{(i)}$. If $a_j(t) = 0$ or $Y_j(t) = 0$, then $X_j(t+1) = X_j(t)$. If $a_i(t) = 1$, then $Y_i(t+1)$ equals one with probability $\theta^1_i(X_i(t), Y_i(t))$,

and zero with probability $1-\theta_i^1(X_i(t),Y_i(t))$. If $a_j(t)=0$, then $Y_j(t+1)$ equals one with probability $\theta_j^0(X_j(t),Y_j(t))$, and zero with probability $1-\theta_j^0(X_j(t),Y_j(t))$.

The stochastic process $(X,Y)=(X(t),Y(t))_{t\geq 0}$ is Markovian; therefore, we only need to consider stationary policies $\pi:\mathcal{S}_1\times\ldots\times\mathcal{S}_N\mapsto\mathcal{A}\triangleq\big\{a\in\{0,1\}^N:a_1+\ldots+a_N=M\big\}$. Denote for every fixed $((x_1,y_1),\ldots,(x_N,y_N))\in\mathcal{S}_1\times\ldots\times\mathcal{S}_N$, the value under a stationary policy π by $J^\pi(((x_1,y_1),\ldots,(x_N,y_N)))$, and it equals

$$\mathbb{E}^\pi\left[\sum_{t=0}^{\infty}\gamma^t\sum_{n=1}^{N}R_n^{a_n(t)}(X_n(t),Y_n(t))\,\middle|\,X_n(0)=x_n,Y_n(0)=y_n,n=1,2,\ldots,N\right],$$

where $a_n(t)=\pi((X_1(t),Y_1(t)),\ldots,(X_N(t),Y_N(t)))$ for every $t\geq 0$. A policy $\pi^*\in\Pi$ is optimal if it maximizes $J^\pi(((x_1,y_1),\ldots,(x_N,y_N)))$ over $\pi\in\Pi$ for every initial state $((x_1,y_1),\ldots,(x_N,y_N))$ in $\mathcal{S}_1\times\ldots\times\mathcal{S}_N$.

3. The Whittle index and indexability

Let us fix an arm and denote its state $(X(t),Y(t))$ as in Section 2, and consider the following auxiliary problem. At each time, the decision maker can either activate the arm or leave it resting. Suppose the current state of the arm is (x,y). If the arm is activated, then reward $R^1(x,y)$ is obtained. If it is rested, then a subsidy $W\in\mathbb{R}$ and the passive reward $R^0(x,y)$ are obtained. The objective is to maximize the expected total discounted reward. Whittle [22] called this problem as the "W-subsidy problem", and it is a variant of the retirement problem (see, for example, Ross [17, Chapter VII]). The so-called Whittle index corresponds to the smallest value of W for which it is optimal to rest the arm. After Whittle index is calculated for every arm, the Whittle index policy is to activate an arm with the largest index. However, this policy makes sense only if any arm which is rested under a subsidy W is also rested under any subsidy $W'>W$. Namely, the set of states at which it is optimal to rest the arm increases monotonically as the value of subsidy W increases. This property is called *indexability*. These concepts were originally introduced by Whittle [23] in the average-reward case, and their counterparts in the discounted case are described by other authors; see, e.g., Niño-Mora [15].

3.1. **Definitions.** Because the indexability and Whittle index are examined for each fixed isolated arm, we omit the subscripts identifying the arm. In the restless bandit problem, a passive arm may change its state and give a nonzero reward. We denote the state process of the arm by the controlled Markov process $S=(S(t))_{t\geq 0}$ on some state space $\mathcal{S}$, the control action at time t by $a(t)$, its transition matrix by

$$p_{ss'}^a=\mathbb{P}\{S(t+1)=s'|S(t)=s,a(t)=a\},\quad s,s'\in\mathcal{S},\ a\in\{0,1\},$$

and the expected reward obtained from the arm under action $a \in \{0,1\}$ by $R^a(s), s \in \mathcal{S}$. The arm is said to be active if action $a = 1$ is taken and passive otherwise.

For every fixed $W \in \mathbb{R}$, the value function of the W-subsidy problem satisfies

$$V(s,W) = \max\left\{L^1(s,W), L^0(s,W)\right\}, \tag{2}$$

where

$$L^1(s,W) = R^1(s) + \gamma \sum_{s' \in \mathcal{S}} p^1_{ss'} V(s',W), \quad L^0(s,W) = W + R^0(s) + \gamma \sum_{s' \in \mathcal{S}} p^0_{ss'} V(s',W)$$

are the maximum expected total rewards if the first action is to activate or to rest the arm, respectively. We let $\Pi(W)$ be the subset of $\mathcal{S}$ in which it is optimal to choose the passive action when the passive subsidy is W. Namely,

$$\Pi(W) \triangleq \left\{s \in \mathcal{S} : L^1(s,W) \leq L^0(s,W)\right\}, \quad W \in \mathbb{R}.$$

If an arm is *indexable* and it is optimal to rest the arm when the value of subsidy is W, then it is also optimal to rest the same arm whenever the value of subsidy is greater than W.

Definition 3.1 (Indexability)**.** *An arm is indexable if* $\Pi(W)$ *is increasing in* W*; namely,*

$$W_1 > W_2 \quad \Longrightarrow \quad \Pi(W_1) \supseteq \Pi(W_2). \tag{3}$$

Definition 3.2 (Whittle index)**.** *The Whittle index of an indexable arm equals*

$$W(s) \triangleq \inf\{W \in \mathbb{R} : s \in \Pi(W)\} \qquad \text{in every state } s \in \mathcal{S}. \tag{4}$$

Under indexability, the Whittle index is the smallest value of W for which both the active and passive actions are optimal.

Definition 3.3 (Whittle index policy)**.** *Suppose that the arms of a restless bandit problem are indexable. The Whittle index policy plays* M *arms with the largest Whittle-indices.*

The W-subsidy problem is a special instance of Problems 1 and 2. If an optimal index policy exists for those problems, then it must also be optimal for W-subsidy problem. This observation will imply the non-existence of an optimal index policy; see Sections 4.5 and 5.3.

4. The Whittle index for Problem 1

This section presents an index policy for Problem 1. We obtain the Whittle index for Problem 1 by studying the W-subsidy problem and prove the non-existence of an optimal index policy. We will see that the Whittle index policy is the unique optimal index policy for the W-subsidy problem up to a monotone transformation. Because the W-subsidy problem is a special case of Problem 1, if there exists an optimal index policy for Problem 1, then it

must be a strict monotone transformation of the Whittle index policy. We prove the non-existence of an optimal index policy by showing an example in which the Whittle index policy is not optimal. We also show that there does not exist an index policy that is uniformly better than our Whittle index policy over all instances of Problem 1.

If an arm is unavailable, then only passive action is available, and the following holds.

Condition 4.1. *For every $x \in \mathcal{X}$, we have*

$$(x,0) \in \Pi(W), \quad W \in \mathbb{R}. \qquad (\inf \varnothing \equiv -\infty) \tag{5}$$

4.1. **The W-subsidy problem for Problem 1.** Recall from Section 1 that $S(t) = (X(t), Y(t))_{t\geq 0}$ is the state process of a potentially unavailable arm. If the current state is $s = (x, y)$ for some $x \in \mathcal{X}$ and $y \in \{0, 1\}$, then the probability that the arm is available at the next stage is given by $\theta^1(x, y)$ if the arm is active, and by $\theta^0(x, y)$ if it is passive; see (1).

For fixed $W \in \mathbb{R}$, let the value function for the corresponding W-subsidy problem be $V((\cdot,\cdot), W) : \mathcal{X} \times \{0, 1\} \mapsto \mathbb{R}$. As in (2), it satisfies the Bellman equation

$$V((x,y),W) = \max\{L^1((x,y),W), L^0((x,y),W)\}, \quad x \in \mathcal{X}, y \in \{0,1\},$$

where

$$L^1((x,y),W) = R^1(x,y) + \gamma \sum_{x' \in \mathcal{X}} p_{xx'} \left[(1-\theta^1(x,y))V((x',0),W) + \theta^1(x,y)V((x',1),W)\right],$$
$$L^0((x,y),W) = W + R^0(x,y) + \gamma \left[\left(1-\theta^0(x,y)\right) V\left((x,0),W\right) + \theta^0(x,y)V\left((x,1),W\right)\right].$$

Let $\mathbb{P}^{1,0}$ be the probability law induced by the policy that the arm is active whenever it is available, and it is passive otherwise. Similarly, let $\mathbb{P}^{0,0}$ be the probability law induced when the arm is always rested. That is, $\mathbb{P}^{1,0}\{X(t+1) = x', Y(t+1) = y' | X(t) = x, Y(t) = y\}$ is

$$\left\{ \begin{array}{ll} p_{xx'} \left[\theta^1(x,1)\right]^{y'} \left[1-\theta^1(x,1)\right]^{1-y'}, & y = 1 \\ \delta_{xx'} \left[\theta^0(x,0)\right]^{y'} \left[1-\theta^0(x,0)\right]^{1-y'}, & y = 0 \end{array} \right\},$$

and $\mathbb{P}^{0,0}\{X(t+1) = x', Y(t+1) = y' | X(t) = x, Y(t) = y\} = \delta_{xx'} \left[\theta^0(x,y)\right]^{y'} \left[1-\theta^0(x,y)\right]^{1-y'}$. Let $\mathbb{E}^{1,0}_{x,y}[\cdot]$ and $\mathbb{E}^{0,0}_{x,y}[\cdot]$ be the expectations with respect to $\mathbb{P}^{1,0}$ and $\mathbb{P}^{0,0}$, respectively, given that $X(0) = x$ and $Y(0) = y$.

Denote by $\rho(x, y)$ the expected total discounted reward from a passive arm whose current state is $(x, y) \in \mathcal{X} \times \{0, 1\}$; namely,

$$\rho(x,y) \triangleq \mathbb{E}^{0,0}_{x,y}\left[\sum_{t=0}^{\infty} \gamma^t R^0(X(t),Y(t))\right] = \mathbb{E}^{0,0}_{x,y}\left[\sum_{t=0}^{\infty} \gamma^t R^0(x,Y(t))\right]. \tag{6}$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$, and $\mathfrak{S}$ be the set of all almost-surely (a.s.) positive stopping times of $(\mathcal{F}_t)_{t\geq 0}$, and define

$$\overline{\mathfrak{S}} \triangleq \{\tau \in \mathfrak{S} : Y(\tau) = 1 \text{ a.s.}\}$$

as the set of positive stopping times at which the arm is available.

The following proposition states that there exists an optimal index policy for the W-subsidy problem. This will be used to obtain the Whittle index and prove the non-existence of an optimal index policy for Problem 1. All of the proofs are deferred to the appendix.

Proposition 4.1. *In the W-subsidy problem, resting the arm is optimal at state $(x,1)$, namely $(x,1) \in \Pi(W)$, if and only if*

$$W \geq (1-\gamma) \sup_{\tau \in \overline{\mathfrak{S}}} \frac{\mathbb{E}_{x,1}^{1,0}\left[\sum_{t=0}^{\tau-1} \gamma^t R^{Y(t)}(X(t),Y(t)) + \gamma^\tau \rho(X(\tau),Y(\tau))\right] - \rho(x,1)}{1 - \mathbb{E}_{x,1}^{1,0}\left[(1-\gamma)\sum_{t=1}^{\tau-1}\gamma^t 1_{\{Y(t)=0\}} + \gamma^\tau\right]}. \tag{7}$$

4.2. **Including passive rewards.** The Whittle index and indexability of Problem 1 are easy consequences of Proposition 4.1. Note that the right-hand side of (7) is the minimum value of subsidy at which it is optimal to make the arm passive when the state of the arm is $(x,1)$; therefore, it corresponds by definition to the Whittle index of the arm.

Proposition 4.2. *For every $x \in \mathcal{X}$ and $y \in \{0,1\}$, the arm is indexable with the index*

$$W(x,y) \triangleq \tag{8}$$
$$\begin{cases} (1-\gamma) \sup_{\tau \in \overline{\mathfrak{S}}} \dfrac{\mathbb{E}_{x,1}^{1,0}\left[\sum_{t=0}^{\tau-1} \gamma^t R^{Y(t)}(X(t),Y(t)) + \gamma^\tau \rho(X(\tau),Y(\tau))\right] - \rho(x,1)}{1 - \mathbb{E}_{x,1}^{1,0}\left[(1-\gamma)\sum_{t=1}^{\tau-1}\gamma^t 1_{\{Y(t)=0\}} + \gamma^\tau\right]}, & \text{if } y = 1, \\ -\infty, & \text{otherwise.} \end{cases}$$

The index in (8) is the generalization of that of Glazebrook et al., [6], who studied a problem where only passive arms may become unavailable, and once arms are unavailable, they never become available. This is a special case of Problem 1 with the following condition.

Condition 4.2. *For every $x \in \mathcal{X}$, suppose that $\theta^1(x,1) = 1$ and $\theta^1(x,0) = \theta^0(x,0) = 0$.*

Corollary 4.1 (Glazebrook et al. [6, Theorem 2])**.** *Suppose Conditions 4.1 and 4.2 hold. Then the arm is indexable with the index $W(x,y)$ defined for every $x \in \mathcal{X}$ by*

$$\begin{cases} (1-\gamma) \sup_{\tau \in \mathfrak{S}} \dfrac{\mathbb{E}_{x,1}^{1,0}\left[\sum_{t=0}^{\tau-1} \gamma^t R^{1}(X(t),Y(t)) + \gamma^\tau \rho(X(\tau),Y(\tau))\right] - \rho(x,1)}{1 - \mathbb{E}_{x,1}^{1,0}\left[\gamma^\tau\right]}, & \text{if } y = 1 \\ -\infty, & \text{otherwise} \end{cases}\Bigg\}. \tag{9}$$

4.3. **The Whittle index if passive arms do not give rewards.** We now derive the Whittle index when passive arms do not give rewards. That is, in addition to Condition 4.1, we also assume the following.

Condition 4.3. *Suppose that $R^0(x,y)=0$ holds for every $x\in\mathcal{X}$ and $y\in\{0,1\}$.*

The arm is indeed indexable, and the Whittle index is given by the following corollary.

Corollary 4.2. *If Conditions 4.1 and 4.3 hold, then the arm is indexable with the index*

(10)
$$W(x,y)=\left\{\begin{array}{ll}\dfrac{(1-\gamma)\sup_{\tau\in\overline{\mathfrak{S}}}\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=1}^{\tau-1}(1-\gamma)\gamma^t 1_{\{Y(t)=0\}}+\gamma^\tau\right]}, & \text{if } y=1\\ -\infty, & \text{otherwise}\end{array}\right\},\quad x\in\mathcal{X}.$$

We omit the proof as the result is immediate from Proposition 4.2 after substituting $R^0(x,y)=0$ for every $x\in\mathcal{X}$ and $y\in\{0,1\}$. The index (10) can be simplified further when the probability of availability does not depend on the state of X and Y.

Corollary 4.3. *Suppose Conditions 4.1 and 4.3 hold, and $\theta^0(x,y)=\theta^1(x,y)=\theta\in[0,1]$ is constant for every $x\in\mathcal{X}$ and $y\in\{0,1\}$. Then the arm is indexable with the index*

(11)
$$W(x,y)=\left\{\begin{array}{ll}\dfrac{(1-\gamma)\sup_{\tau\in\overline{\mathfrak{S}}}\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\gamma(1-\theta)-\theta\mathbb{E}^{1,0}_{x,1}[\gamma^\tau]}, & \text{if } y=1\\ -\infty, & \text{otherwise}\end{array}\right\},\quad x\in\mathcal{X}.$$

4.4. **The convergence of the Whittle index to the Gittins index or to the immediate expected reward.** We now analyze the Whittle index (11) as a function of probability of availability $\theta\in[0,1]$. Firstly, the Whittle index is a generalization of the Gittins index: they coincide when the arm is always available, as the next corollary shows; therefore, they are optimal for the classical bandit problem.

Corollary 4.4. *Suppose the arm is always available; i.e.,*

$$\theta^0(x,y)=\theta^1(x,y)=1,\quad x\in\mathcal{X},\ y\in\{0,1\}. \tag{12}$$

Then the arm is indexable with the index

$$W(x,y)=\left\{\begin{array}{ll}\sup\limits_{\tau\in\mathfrak{S}}\dfrac{\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right]}{\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t\right]}, & \text{if } y=1\\ -\infty, & \text{otherwise}\end{array}\right\},\quad x\in\mathcal{X}. \tag{13}$$

Since each arm is available with probability one, the probability law $\mathbb{P}^{1,0}$ chooses the active action all the time. Thus, the Whittle index policy is optimal in the classical bandit problem.

Secondly, if $\theta^0(x,y) = \theta^1(x,y) = 0$ for every $x \in \mathcal{X}$ and $y \in \{0,1\}$, then the index becomes

$$W(x,1) = R^1(x,1), \quad x \in \mathcal{X}. \tag{14}$$

The next proposition shows that the limits of the Whittle index in (11) as $\theta \nearrow 1$ and $\theta \searrow 0$ exist and coincide with its extreme values in (13) and (14), respectively.

Proposition 4.3. *The Whittle index $W(x,1)$ defined in (11) converges to the Gittins index as $\theta \nearrow 1$, and to the one-time reward $R^1(x,1)$ as $\theta \searrow 0$ uniformly over $x \in \mathcal{X}$.*

The result is also intuitive. If the probability of availability is small, then the decision maker becomes myopic; she does not care too much about the future rewards, because she expects that the arm will not be available for most of the time. Consider, for example, the situation in which there is one arm which is always available and gives her a constant reward while the other arms are available today, but she knows that they will not be available in the future. The optimal strategy is to pick today the arm with the highest immediate reward and to pick tomorrow and in the future the only arm that is available. On the other hand, as $\theta \nearrow 1$, the problem becomes more similar to the classical bandit problem, and the index converges to the optimal Gittins index by Proposition 4.3.

4.5. **The non-existence of an optimal policy.** We conclude this section by proving the non-existence of an optimal index policy for Problem 1. This is true even in a special case in which $M = 1$, passive arms do not give rewards, and the probability of availability is consistent. We claim that if there exists an optimal index policy, then it must be a strict monotone transformation of the index defined by (8). We then show the non-existence of an optimal index policy by showing an example in which the index policy is not optimal.

Proposition 4.4. *Any index that is optimal for every instance of Problem 1 must be a strict monotone transformation of the index $W(\cdot,\cdot)$ in (8).*

Proposition 4.5. *An optimal index policy does not exist for Problem 1.*

If an optimal index policy exists for Problem 1, then it must be a strict monotone transformation of (8) by Proposition 4.4. For, otherwise, it will not be optimal for the W-subsidy problem, a special case of Problem 1. This also means that there does not exist an index policy that uniformly dominates Whittle index policy over all of the instances of Problem 1.

5. The Whittle index for Problem 2

Unlike in the previous problem, here we assume that the active action is still available even when the arm is unavailable. Activating the arm is equivalent to repairing the arm. We denote by $C_n(x) > 0$ the repair cost when arm n is broken at state x. If a broken arm is not repaired, then it will be unavailable (or stay broken) the next time with probability one. We assume that passive arms do not give rewards, and that the reward obtained from activating an available arm is positive as in the following condition.

Condition 5.1. *For every $x \in \mathcal{X}$, $y \in \{0,1\}$, and $n = 1, \dots, N$, suppose that $R_n^1(x,1) \geq 0$, $R_n^1(x,0) = -C_n(x) < 0$, $R_n^0(x,y) = 0$, and $\theta_n^0(x,0) = 0$.*

5.1. **The W-subsidy problem and the Whittle index for Problem 2.** As for Problem 1, we first consider the W-subsidy problem for Problem 2. Under Condition 5.1, the value function of the W-subsidy problem satisfies

$$V((x,y),W) = \max\{L^1((x,y),W), L^0((x,y),W)\},$$

where for every $x \in \mathcal{X}$, we have

$$L^1((x,1),W) = R^1(x,1) + \gamma \sum_{x' \in \mathcal{X}} p_{xx'} \left[(1-\theta^1(x,1))V((x',0),W) + \theta^1(x,1)V((x',1),W)\right],$$
$$L^0((x,1),W) = W + \gamma \left[(1-\theta^0(x,1))V((x,0),W) + \theta^0(x,1)V((x,1),W)\right],$$

and

$$L^1((x,0),W) = -C(x) + \gamma \left[(1-\theta^1(x,0))V((x,0),W) + \theta^1(x,0)V((x,1),W)\right],$$
$$L^0((x,0),W) = W + \gamma V((x,0),W).$$

Similar to $\mathbb{P}^{1,0}$ and $\mathbb{P}^{0,0}$ of the previous section, let $\mathbb{P}^{1,1}$ be the probability law induced by the policy that activates the arm forever and $\mathbb{E}^{1,1}$ denote the expectation with respect to $\mathbb{P}^{1,1}$. Moreover, let $\psi(x)$ be the expected total discounted reward if the arm is active forever starting in state $(x,1)$ at time zero. Namely,

$$\psi(x) \triangleq \mathbb{E}_{x,1}^{1,1}\left[\sum_{t=0}^{\infty} \gamma^t \left\{R^1(X(t),Y(t))1_{\{Y(t)=1\}} - C(X(t))1_{\{Y(t)=0\}}\right\}\right]. \tag{15}$$

Condition 5.2. *Suppose that we have $\psi(x) \geq -C(x)/(1-\gamma)$ for every $x \in \mathcal{X}$.*

Condition 5.2 is satisfied, for example, if

(i) the arm never breaks under the active action; i.e., $\theta^1(x,1) = 1$ for every $x \in \mathcal{X}$, or
(ii) $C(X(t))$ is constant or non-increasing almost surely under the active action.

Proposition 5.1. *Under Condition 5.2, in the W-subsidy problem for Problem 2, activating the arm is optimal if and only if*

$$W \geq (1-\gamma) \sup_{\tau \in \mathfrak{S}} \frac{\mathbb{E}_{x,y}^{1,1}\left[\sum_{t=0}^{\tau-1} \gamma^t R^1(X(t), Y(t))\right]}{1 - \mathbb{E}_{x,y}^{1,1}\left[\gamma^\tau\right]}.$$

Proposition 5.2. *If Condition 5.2 holds, then the arm is indexable with the index*

$$W(x,y) = (1-\gamma) \sup_{\tau \in \mathfrak{S}} \frac{\mathbb{E}_{x,y}^{1,1}\left[\sum_{t=0}^{\tau-1} \gamma^t R^1(X(t), Y(t))\right]}{1 - \mathbb{E}_{x,y}^{1,1}\left[\gamma^\tau\right]}, \quad x \in \mathcal{X}, y \in \{0,1\}. \tag{16}$$

Remark 5.1. *If Condition 4.2 holds, then the problem reduces to a problem studied by Glazebrook et al.* [6] *where passive arms do not give rewards, and the indices coincide.*

5.2. **The connection to the bandit problems with switching costs.** The problem is reduced to the classical bandit problem with switching costs if $\theta^1(x,1) = 1$, and if $C_n(x)$ is the cost of switching to arm n when the current state of arm n is x. Condition 5.2 holds because $\theta^1(x,1) = 1$; therefore, the Whittle index is indexable. Glazebrook et al. [9] have formulated the problem as a restless bandit problem. Their formulation is slightly different from ours in that one does not have to wait for the arm to be fixed; when one plays a broken arm, he obtains its immediate reward minus the switching cost and the arm is guaranteed to be available the next time. Nevertheless, the form of their index is the same as ours. They showed numerically that their index policy is nearly optimal.

5.3. **The non-existence of an optimal policy.** As in Problem 1, an optimal index policy does not exist for Problem 2. We again show that if there exists one, then the corresponding index must be a strict monotone transformation of our index, and we give an example where our index policy is not optimal. Proof of Proposition 5.3 is omitted because it is very similar to that of Proposition 4.4.

Proposition 5.3. *If there is an optimal index policy for Problem 2, then the index must be a strict monotone transformation of the index $W(\cdot,\cdot)$ in (16).*

Proposition 5.4. *An optimal index policy does not exist for Problem 2.*

6. The restart-in problem

We have developed the Whittle indices for Problems 1 and 2 in the previous sections. Here we assume that passive arms do not give rewards and discuss how to compute the indices in (10) and (16). To this end, we develop the *restart-in problem* representation of the indices. The restart-in problem representation of the Gittins index for the classical multi-armed bandit problem was introduced by Katehakis and Veinott [13]. The index in (16) is

similar to the Gittins index; therefore, we first formulate it as a restart-in problem. We then propose a generalization of restart-in problem representation for the Whittle index in (10).

6.1. **The restart-in problem for the Gittins index.** We first review the restart-in problem formulated by Katehakis and Veinott [13]. Consider the classical multi-armed bandit problem where a fixed arm follows a stochastic process $S = (S(t))_{t\geq 0}$ on some state space $\mathcal{S}$ and evolves according to a transition matrix $p = (p_{ss'})_{ss'\in\mathcal{S}}$ under the active action, and let $R(s)$ be the one-time reward obtained when the state of the arm is s and when it is active.

Katehakis and Veinott [13] showed that the Gittins index of the arm at state $\tilde{s} \in \mathcal{S}$ equals $(1-\gamma)\nu_{\tilde{s}}$, where $\nu = (\nu_s)_{s\in\mathcal{S}}$ satisfies the optimality equation

$$\nu_s = \max\left(R(s) + \gamma \sum_{s'\in\mathcal{S}} p_{ss'}\nu_{s'},\ R(\tilde{s}) + \gamma \sum_{s'\in\mathcal{S}} p_{\tilde{s}s'}\nu_{s'} \right), \quad s \in \mathcal{S}; \tag{17}$$

particularly,

$$\nu_{\tilde{s}} = \sup_{\tau>0} \frac{\mathbb{E}\left[\sum_{t=0}^{\tau-1}\gamma^t R(S(t))\middle| S(0)=\tilde{s}\right]}{\mathbb{E}\left[1-\gamma^\tau\right]} = \sup_{\tau>0} \frac{\mathbb{E}\left[\sum_{t=0}^{\tau-1}\gamma^t R(S(t))\middle| S(0)=\tilde{s}\right]}{(1-\gamma)\mathbb{E}\left[\sum_{t=0}^{\tau-1}\gamma^t\right]}.$$

The Gittins index at a fixed state $\tilde{s}$ is the solution of $|\mathcal{S}|$ equations defined by (17), and it can be calculated by the value-iteration algorithm applied to (17).

We now characterize the Whittle indices in (5.2) and (10) of a potentially unavailable arm as the value function of a restart-in problem.

6.2. **The restart-in problem representation of (16).** Because (16) and Gittins index are similar, we can use the restart-in problem representation of the Gittins index, after we multiply each one-time reward by $(1-\gamma)$. Namely, $W(\tilde{x},\tilde{y})$ equals $\nu_{\tilde{x},\tilde{y}}$ if $R^1(x,0) = -C(x)$ and $(\nu_{x,y})_{x\in\mathcal{X},y\in\{0,1\}}$ satisfies for every $x \in \mathcal{X}$ and $y \in \{0,1\}$,

$$\begin{aligned}\nu_{x,y} = \max\Bigg(& (1-\gamma)R^1(x,y) + \gamma\sum_{x'\in\mathcal{X}} p_{xx'}\left[\theta^1(x,y)\nu_{x',1} + (1-\theta^1(x,y))\nu_{x',0}\right], \\ & (1-\gamma)R^1(\tilde{x},\tilde{y}) + \gamma\sum_{x'\in\mathcal{X}} p_{\tilde{x}x'}\left[\theta^1(\tilde{x},\tilde{y})\nu_{x',1} + (1-\theta^1(\tilde{x},\tilde{y}))\nu_{x',0}\right]\Bigg).\end{aligned} \tag{18}$$

6.3. **The restart-in problem representation for (10).** Fix $\tilde{x} \in \mathcal{X}$ and define $(\nu_{x,y})_{x\in\mathcal{X},y\in\{0,1\}}$ such that, for every $x \in \mathcal{X}$,

$$\begin{aligned}\nu_{x,1} = \max\Bigg(& (1-\gamma)R^1(x,1) + \gamma\sum_{x'\in\mathcal{X}} p_{xx'}\left[\theta^1(x,1)\nu_{x',1} + (1-\theta^1(x,1))\nu_{x',0}\right], \\ & (1-\gamma)R^1(\tilde{x},1) + \gamma\sum_{x'\in\mathcal{X}} p_{\tilde{x}x'}\left[\theta^1(\tilde{x},1)\nu_{x',1} + (1-\theta^1(\tilde{x},1))\nu_{x',0}\right]\Bigg),\end{aligned} \tag{19}$$

and

$$\nu_{x,0} = (1-\gamma)\nu_{\tilde{x},1} + \gamma\left(\theta^0(x,0)\nu_{x,1} + (1-\theta^0(x,0))\nu_{x,0}\right). \tag{20}$$

Next proposition shows that if $(\nu_{x,y})_{x\in\mathcal{X},y\in\{0,1\}}$ satisfies (19) and (20), then $\nu_{\tilde{x},1}$ coincides with the index in (2) of the arm in state $(\tilde{x},1)$.

Proposition 6.1. *If $(\nu_{x,y})_{x\in\mathcal{X},y\in\{0,1\}}$ satisfies (19) and (20), then we have*

$$\nu_{\tilde{x},1} = (1-\gamma)\sup_{\tau\in\overline{\mathfrak{S}}} \frac{\mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=1}^{\tau-1}(1-\gamma)\gamma^t 1_{\{Y(t)=0\}}+\gamma^\tau\right]}. \tag{21}$$

Remark 6.1. *Suppose $\theta^1(x,1)=1$ for every $x\in\mathcal{X}$ as in the classical multi-armed bandit problem. Then (19) becomes*

$$\nu_{x,1} = \max\left\{(1-\gamma)R^1(x,1) + \gamma\sum_{x'\in\mathcal{X}} p_{xx'}\nu_{x',1},\ (1-\gamma)R^1(\tilde{x},1) + \gamma\sum_{x'\in\mathcal{X}} p_{\tilde{x}x'}\nu_{x',1}\right\},$$

which is Katehakis and Veinott's [13] *restart-in problem representation of the Gittins index after multiplication by $(1-\gamma)$. Indeed, $\nu_{\tilde{x},1}$ equals the Gittins index of the arm in state $(\tilde{x},1)$.*

7. An example

We will illustrate how to evaluate Whittle indices in (10) and (16) and compare Whittle and other index policies for Problems 1 and 2, in which the reward of each arm is a Bernoulli random variable with some unknown success probability. The success probability of arm n is a random variable λ_n having beta distribution with parameters a_n and b_n; namely,

$$\mathbb{P}\{\lambda_n \in dr\} = \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)} r^{a_n-1}(1-r)^{b_n-1}dr,$$

and let $C_n > 0$ be the repair cost for arm n in Problem 2.

Katehakis and Derman [12] approximate the Gittins index for the classical two-armed bandit problem with Bernoulli distributed rewards. We generalize their approximation to calculate the Whittle indices in (10) and (16). In the next section, we compute the Whittle indices and evaluate corresponding index policies. Here we shall omit the subscripts that identify the arms, because we compute the index for each arm separately.

If the parameters of the current beta distribution of λ are (a,b), and if c successes and d failures are observed after $c+d$ plays, then the posterior probability distribution of λ is also beta with parameters $(a+c,b+d)$.

Let $X(t)$ denote the parameters of the posterior beta distribution of λ after t plays. Then the pair (X,Y) stores all of the information needed; it is a Markov process that evolves

under the active action according to transition probabilities

$$p_{(a,b),(a+1,b)} = \frac{a}{a+b}, \qquad p_{(a,b),(a,b+1)} = \frac{b}{a+b}.$$

The one-time expected reward given $X(t) = (a,b)$ and $Y(t) = 1$ at time t is $a/(a+b)$. We formulate the restart-in problem representation for Problems 1 and 2 and describe a computational method analogous to that of Katehakis and Derman [12].

7.1. **The restart-in problem for Problem 1.** Specializing (19) and (20), the Whittle index of the arm in state $(x,1) = ((\tilde{a},\tilde{b}),1)$ is given by $W((\tilde{a},\tilde{b}),1) = \nu_{(\tilde{a},\tilde{b}),1}$, which satisfy

$$\nu_{(a,b),1} = \max(\mathcal{L}(a,b), \mathcal{L}(\tilde{a},\tilde{b})), \tag{22}$$

$$\nu_{(a,b),0} = (1-\gamma)\nu_{(\tilde{a},\tilde{b}),1} + \gamma\left[\theta^0((a,b),0)\nu_{(a,b),1} + (1-\theta^0((a,b),0))\nu_{(a,b),0}\right], \tag{23}$$

for every $(a,b) \in \mathbb{N}^2$, where

$$\begin{aligned}\mathcal{L}(a,b) &= (1-\gamma)R^1((a,b),1) + \gamma\left[\theta^1((a,b),1)\left(p_{(a,b)(a+1,b)}\nu_{(a+1,b),1} + p_{(a,b)(a,b+1)}\nu_{(a,b+1),1}\right)\right.\\ &\quad \left.+(1-\theta^1((a,b),1))\left(p_{(a,b)(a+1,b)}\nu_{(a+1,b),0} + p_{(a,b)(a,b+1)}\nu_{(a,b+1),0}\right)\right]\\ &= (1-\gamma)\frac{a}{a+b} + \gamma\left[\theta^1((a,b),1)\left(\frac{a}{a+b}\nu_{(a+1,b),1} + \frac{b}{a+b}\nu_{(a,b+1),1}\right)\right.\\ &\quad \left.+(1-\theta^1((a,b),1))\left(\frac{a}{a+b}\nu_{(a+1,b),0} + \frac{b}{a+b}\nu_{(a,b+1),0}\right)\right].\end{aligned}$$

7.2. **The restart-in problem for Problem 2.** The Whittle index of the arm at state $(\tilde{x},\tilde{y}) = ((\tilde{a},\tilde{b}),\tilde{y})$ is given by $W((\tilde{a},\tilde{b}),\tilde{y}) = \nu_{(\tilde{a},\tilde{b}),\tilde{y}}$, which satisfy

$$\nu_{(a,b),y} = \max(\mathcal{L}((a,b),y), \mathcal{L}((\tilde{a},\tilde{b}),\tilde{y})) \tag{24}$$

for every $(a,b) \in \mathbb{N}^2$ and $y \in \{0,1\}$, where $\mathcal{L}((a,b),y)$ equals

$$\begin{aligned}&(1-\gamma)R^1((a,b),y) + \gamma\left[\theta^1((a,b),y)\left(p_{(a,b)(a+1,b)}\nu_{(a+1,b),1} + p_{(a,b)(a,b+1)}\nu_{(a,b+1),1}\right)\right.\\ &\qquad \left.+(1-\theta^1((a,b),y)\left(p_{(a,b)(a+1,b)}\nu_{(a+1,b),0} + p_{(a,b)(a,b+1)}\nu_{(k,l+1),0}\right)\right]\\ &= (1-\gamma)\left(\frac{a}{a+b}1_{\{1\}}(y) - C1_{\{0\}}(y)\right) + \gamma\left[\theta^1((a,b),y)\left(\frac{a}{a+b}\nu_{(a+1,b),1} + \frac{b}{a+b}\nu_{(a,b+1),1}\right)\right.\\ &\left.+(1-\theta^1((a,b),y)\left(\frac{a}{a+b}\nu_{(a+1,b),0} + \frac{b}{a+b}\nu_{(a,b+1),0}\right)\right].\end{aligned}$$

7.3. **Computing the Whittle indices.** We can calculate the Whittle indices (10) and (16) by solving, with the value iteration algorithm, the Bellman equations (22), (23), and (24), respectively. Katehakis and Derman [12] calculated the Gittins index for a classical bandit problem with two Bernoulli arms after truncating the state space of the process X to

$$Z_L = \left\{(a,b) \in \mathbb{N}^2 : a+b \le L\right\} \tag{25}$$

for some fixed integer $L > 0$. In the same manner, we only consider states $s = (x, y)$ where $x = (a, b) \in Z_L$ and $y \in \{0, 1\}$. Katehakis and Derman [12] proved that as L increases, their approximation converges to the true value of the Gittins index. It is easy to prove that the same result holds in our settings.

7.4. **Some bounds on the optimal value function of the restart-in problem.** We discuss how to obtain some upper and lower bounds on the optimal value function. In principle, the optimal values can be computed by solving the dynamic programming equations using, for example, the value iteration algorithm. In practice, however, they are computationally intractable when the number of arms is large.

Let $J^*(s)$ denote the optimal value when the state of arm n is $s_n = ((a_n, b_n), y_n)$ for $n = 1, \ldots, N$. Let $\overline{j}$ and $\underline{j}$ be some upper and lower bounds of J^*, respectively. Namely,

$$\underline{j}(s) \leq J^*(s) \leq \overline{j}(s), \quad s \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_N.$$

We truncate the state space to $(Z_L \times \{0, 1\})^N$, where Z_L is defined as in (25).

Now we can replace the reward at the boundaries by their bounds and use backward induction to obtain bounds on the value function on the entire truncated state space. For $s \in (Z_L \times \{0, 1\})^N$, define $\overline{J}(s) = \overline{j}(s)$ and $\underline{J}(s) = \underline{j}(s)$ where $\overline{Z}_L = \{(a, b) \in \mathbb{N}^2 : a+b = L\}$, and calculate $\overline{J}(s)$ and $\underline{J}(s)$ by solving the same Bellman equation for $J^*(\cdot)$. By monotonicity,

$$\underline{J}(s) \leq J^*(s) \leq \overline{J}(s), \quad s \in (Z_L \times \{0, 1\})^N .$$

8. Numerical results for Problem 1

We provide numerical results obtained for Problem 1 in this section. We considered the problem where the reward process is defined as in Section 7, and the probability that an arm is available is constant. The corresponding Whittle index is given by (11).

8.1. **Computing the Whittle indices.** We computed the Whittle indices for probabilities of availability $\theta = 0.1, 0.3, 0.5, 0.7, 0.9, 1.0$ and $\gamma = 0.9$ using the algorithm described in the previous section with $L = 200$. These indices are tabulated in Appendix B. Recall that the Whittle index is a generalization of the Gittins index and coincides with the Gittins index when $\theta = 1$. Indeed, we obtained the same values as Katehakis and Derman [13] for $\theta = 1$ after multiplying by $(1 - \gamma)$. Moreover, as the value of θ decreases to 0, as we proved in Proposition 4.3, the index converges to the one-time reward. Note that the indices are very close to the one-time reward $a/(a + b)$ when $\theta = 0.1$.

θ_1	θ_2	θ_3	lower/upper bounds	Whittle (95 % CI)	Gittins (95 % CI)
1.0	1.0	1.0	(6.49292, 6.60618)	6.5426 (6.5381, 6.5471)	6.5426 (6.5381, 6.5471)
0.7	0.7	1.0	(6.17190, 6.22723)	6.1782 (6.1737, 6.1827)	6.1673 (6.1630, 6.1716)
0.7	0.5	1.0	(6.04527, 6.09630)	6.0304 (6.0259, 6.0349)	6.0357 (6.0314, 6.0400)
0.7	0.3	1.0	(5.92097, 5.97950)	5.9295 (5.9248, 5.9342)	5.9076 (5.9033, 5.9119)
0.7	0.1	1.0	(5.81247, 5.88546)	5.8155 (5.8106, 5.8204)	5.7866 (5.7821, 5.7911)
0.5	0.5	1.0	(5.89183, 5.93610)	5.8942 (5.8897, 5.8987)	5.8826 (5.8783, 5.8869)
0.5	0.3	1.0	(5.74100, 5.79254)	5.7308 (5.7261, 5.7355)	5.7293 (5.7248, 5.7338)
0.5	0.1	1.0	(5.60574, 5.67005)	5.5987 (5.5938, 5.6036)	5.5868 (5.5821, 5.5915)
0.3	0.3	1.0	(5.55919, 5.62343)	5.5503 (5.5454, 5.5552)	5.5562 (5.5517, 5.5607)
0.3	0.1	1.0	(5.39082, 5.39082)	5.3924 (5.3871, 5.3977)	5.3864 (5.3815, 5.3913)
0.1	0.1	1.0	(5.18692, 5.31909)	5.1898 (5.1843, 5.1953)	5.1888 (5.1837, 5.1939)

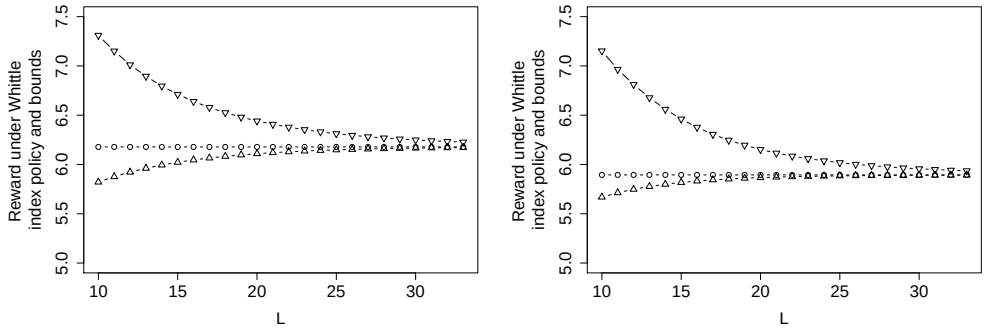
TABLE 1. The comparison of the optimal policy and the Whittle and Gittins index policies

8.2. **Evaluation of the Whittle index policy.** We start by comparing the Whittle index policy and the optimal policy. Suppose that there are exactly three arms. Arms 1 and 2 are available with some fixed probability $\theta \in [0, 1]$, and arm 3 is available with probability one. With larger number of arms, an interesting and hopefully tight lower bound of the optimal value can be obtained by applying the Gittins index. Analogous to the Whittle index policy, we define the Gittins index policy

Definition 8.1 (Gittins index policy)**.** *The Gittins index policy chooses, by definition, the arm with the largest Gittins index among the available arms. The Gittins index coincides with the Whittle index when* $\theta = 1$.

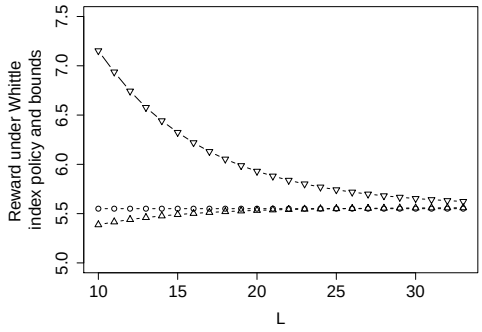
When the number of arms is more than three, we compared the Whittle index policy with the Gittins index policy. We have tested several cases in which M out of N arms are played each time. The parameters of the initial beta distribution of the reward from each arm is $(a, b) = (1, 1)$, and there are always M arms with $\theta = 1$. In case of a tie, we pick a random subset of M arms with the largest indices.

8.3. **Results.** Table 1 compares the upper and lower bounds on the optimal expected total discounted reward values and expected total discounted rewards obtained by applying the Whittle and Gittins index policies when the number of arms is three. The bounds on the optimal values were obtained by the backward induction algorithm as described in Section 7.4 with $\overline{j}(s) = \sum_{t=0}^{\infty} \gamma^t 1 = 1/(1-\gamma)$, and $\underline{j}(s) = a_3/(a_3 + b_3)$. Here, $\overline{j}(s) \geq J^*(s)$ because one-time reward is bounded from above by one, and $\underline{j}(s) \leq J^*(s)$ because $\underline{j}(s)$ can be

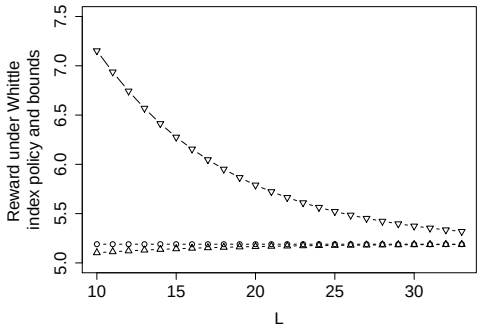


(a) $\theta_1 = 0.7, \theta_2 = 0.7, \theta_3 = 1.0$

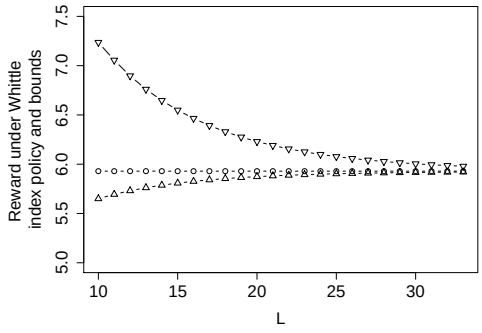
(b) $\theta_1 = 0.5, \theta_2 = 0.5, \theta_3 = 1.0$



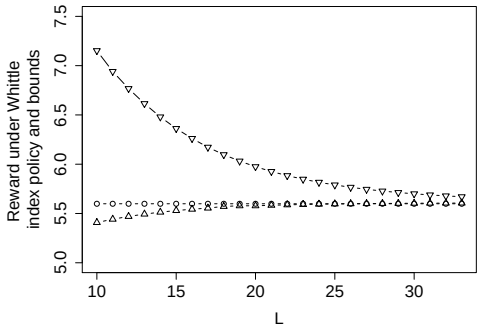
(c) $\theta_1 = 0.3, \theta_2 = 0.3, \theta_3 = 1.0$



(d) $\theta_1 = 0.1, \theta_2 = 0.1, \theta_3 = 1.0$



(e) $\theta_1 = 0.7, \theta_2 = 0.3, \theta_3 = 1.0$



(f) $\theta_1 = 0.5, \theta_2 = 0.1, \theta_3 = 1.0$

FIGURE 1. The upper/lower bounds on the optimal expected total discounted reward and expected total discounted reward under the Whittle index policy as a function of L at state $(X_1, Y_1) = (X_2, Y_2) = (X_3, Y_3) = ((1, 1), 1)$.

attained by pulling arm 3, which is always available. The values under the Whittle and Gittins index policies are calculated by Monte Carlo simulation based on 1,000,000 samples. Figure 1 shows the upper and lower bounds and the value under the Whittle index policy as a function of L used in the definition of the truncated space Z_L in (25). The bounds converge to the optimal value as L increases, and we can see how close the value obtained by the Whittle index policy is to the optimal value.

N	M	Whittle (95% CI)	Gittins (95% CI)	M	Whittle (95% CI)	Gittins (95% CI)
Case 1:						
2	1	5.5474 (5.5313, 5.5635)	5.5587 (5.543, 5.5744)			
6	1	6.6181 (6.6056, 6.6306)	6.4784 (6.4672, 6.4896)			
12	1	6.9357 (6.9245, 6.9464)	6.5600 (6.5502, 6.5698)	2	13.7052 (13.6885, 13.7219)	13.3439 (13.3288, 13.3590)
18	1	7.0225 (7.0119, 7.0331)	6.6442 (6.6342, 6.6542)	3	20.6722 (20.6518, 60.6926)	20.1230 (20.1052, 20.1408)
24	1	7.0307 (7.0202, 7.0410)	6.6391 (6.6291, 6.6491)	4	27.6446 (27.6213, 27.6679)	26.8526 (26.8526, 26.8934)
30	1	7.0301 (7.0197, 7.0405)	6.6459 (6.6359, 6.6559)	5	34.6193 (34.5934, 34.6452)	33.6254 (33.6027, 33.6481)
36	1	7.0297 (7.0193, 7.0401)	6.6346 (6.6246, 6.6446)	6	41.5967 (41.5687, 41.6254)	40.3890 (40.3643, 40.4137)
Case 2:						
3	1	5.9132 (5.8985, 5.9279)	5.9255 (5.9116, 5.9394)			
6	1	6.6227 (6.6102, 6.6352)	6.4724 (6.4612, 6.4836)			
12	1	6.9309 (6.9199, 6.9419)	6.5544 (6.5446, 6.5642)	2	13.5260 (13.5088, 13.5432)	13.1447 (13.1298, 13.1596)
18	1	6.9930 (6.9824, 7.0036)	6.5223 (6.5127, 6.5319)	3	20.4339 (20.4129, 20.4549)	19.7741 (19.7563, 19.7919)
24	1	7.0178 (7.0072, 7.0284)	6.4806 (6.4708, 6.4904)	4	27.3402 (27.3163, 27.3641)	26.4494 (26.4290, 26.4698)
30	1	7.0214 (7.0110, 7.0318)	6.4530 (6.4432, 6.4628)	5	34.2210 (34.1943, 34.2477)	33.0992 (33.0765, 33.1219)
36	1	7.0266 (7.0162, 7.037)	6.4499 (6.4401, 6.4597)	6	41.1573 (41.1283, 41.1863)	39.7806 (39.7561, 39.8051)
Case 3:						
6	1	6.5085 (6.4958, 6.5212)	6.3575 (6.3459, 6.3691)			
12	1	6.8716 (6.8604, 6.8828)	6.5676 (6.5574, 6.5778)	2	13.3253 (13.3079, 13.3427)	12.9986 (12.9829, 13.0143)
18	1	6.9434 (6.9326, 6.9542)	6.5944 (6.5842, 6.6046)	3	20.1669 (20.1457, 20.1881)	19.6288 (19.6098, 19.6478)
24	1	6.9805 (6.9699, 6.9911)	6.6165 (6.6065, 6.6265)	4	27.0206 (26.9963, 27.0449)	26.2567 (26.2350, 26.2786)
30	1	7.0076 (6.9970, 7.0182)	6.6249 (6.6149, 6.6349)	5	33.8775 (33.8504, 33.9044)	32.8960 (32.8717, 32.9203)
36	1	7.0178 (7.0071, 7.0283)	6.6399 (6.6299, 6.6499)	6	40.7183 (40.6842, 40.7434)	39.5345 (39.5079, 39.5609)

TABLE 2. The comparison of the expected total discounted rewards under Whittle and Gittins index policies.

For larger number of arms, we compared the Whittle and Gittins index policies in the following three cases, where M out of N arms are played every time; the results are shown in Table 2.

case 1: $N/2$ arms are available with probability $\theta = 1.0$ and 0.5,
case 2: $N/3$ arms are available with probability $\theta = 1.0$, 0.7, and 0.3,
case 3: $N/6$ arms are available with probability $\theta = 1.0$, 0.9, 0.7, 0.5, 0.3, and 0.1.

As seen from Table 1 and Figure 1, both the Whittle and Gittins index policies are as good as the optimal policy, at least when the number of arms is small.

The Whittle index policy outperforms the Gittins index policy in most of the examples. The Gittins index policy does not utilize the likelihood of each arm's future availability, but the Whittle index takes that information into account, and therefore it does better than the Gittins index policy. Nevertheless, the Gittins index policy should give tight lower bounds considering that the policy is optimal when each arm is always available.

N	M	optimal values (95% CI)
1	1	5.0021 (4.9962, 5.0080)
2	1	6.0963 (6.0912, 6.1014)
5	1	6.8722 (6.8602, 6.8842)
20	1	7.0151 (7.0045, 7.0257)
40	1	7.0261 (7.0156, 7.0364)
80	1	7.0256 (7.0152, 7.0360)

TABLE 3. The approximation to the optimal value if all the arms are always available.

Consider a multi-armed bandit problem with large number of initially identical arms where each arm is always available. Its optimal value can be used as a upper bound on the optimal values of the problems in Table 2. We show in Table 3 the approximation of the optimal values when the initial state of each arm is $(a, b) = (1, 1)$ and $\theta = 1$, calculated by Monte Carlo simulation and by using the Gittins indices. Note that the value converges as $N \to \infty$, which gives a rough upper bound of the optimal value, around 7.026 in this example.

The near-optimality of the Whittle index when N is large can be observed in Table 2 by comparing the values with the rough upper bound 7.026. When $M = 1$, the Whittle index policy gets close to be optimal as N increases, because the number of arms having $\theta = 1$ increases with N, and the Whittle index policy tends to choose the arms with $\theta = 1$; therefore, the problem gets closer to a classical multi-armed bandit problem, and the Whittle index policy acts like the Gittins index policy as long as it chooses the arms with $\theta = 1$. When $M > 1$, the Whittle index policy is still strong as the value of the policy is very close to the upper bound per arm, 7.026. In this way, the numerical results shown in this section show a strong performance of the Whittle index policy.

9. Numerical results for Problem 2

We evaluate the Whittle index policy for Problem 2 in a similar manner. We again consider the problem with the reward process defined in Section 7 but with a simplification so that the optimal value can be computed when the number of arms is small: the probability that an arm is available does not depend on the state of X or whether or not it is active; namely,

$$\theta_n^1(x, 1) = \theta_n^0(x, 1) = \theta_n(1),\ \theta_n^1(x, 0) = \theta_n(0),\ \theta_n^0(x, 0) = 0, \quad x \in \mathcal{X},\ \ n = 1, \ldots, N,$$

for some $\theta_n(1)$ and $\theta_n(0)$. We evaluate the Whittle index policy in Problem 2 and compare it with the optimal policy when the number of arms is two, and with Gittins-like index policies defined later in this section. We focus on the case $M = 1$. The Whittle indices are calculated using the algorithm described in Section 7 and are listed in Appendix C.

The problem will be more practical if the controller has an option to retire, because when all arms are broken he may think that it is not worth fixing any of the arms. We introduce a dummy arm, type 0 arm, which is initially broken and whose repair cost is zero, and it always breaks down immediately after every repair (i.e. $\theta^1(x,0) = 0$). Its Whittle index is always zero. Choosing this arm is equivalent to retiring; i.e., collecting 0 reward, in the remainder. In this way, the retirement option can be added to Problem 2.

We first compare Gittins and Whittle index policies for Problem 2. Define Policy 1 and Policy 2 such that

(i) Policy 1 always chooses the arm with the largest Gittins index among the available arms. It retires if arms are broken.
(ii) Policy 2 always chooses the arm with the largest Gittins index among the available arms. If no arm is available, then it chooses the arm with the largest Gittins index pretending that all arms are available.

The Gittins index is calculated regardless of the value of the break-down probability or the repair cost of the arm. The indices are defined only when the arm is available.

We expect Policies 1 and 2 to work reasonably well because they are optimal when arms never break down. Policy 1 is antithetical to Policy 2 in that the former is pessimistic while the latter is optimistic about repairing arms. In this problem, there is always a trade-off between repairing and ignoring broken arms. We expect that the Whittle index policy resolves this trade-off.

The initial prior of each arm is $(a,b) = (1,1)$, and each arm is initially available; i.e., $Y(0) = 1$. The upper and lower bounds on the optimal values when the number of arms is two are obtained by the backward induction algorithm (see Section 7.4) starting with $\overline{j}(s) = 1/(1-\gamma)$ and $\underline{j}(s) = 0$, respectively, where the lower bound $\underline{j}(\cdot)$ can be attained by retiring (or by pulling forever type 0 arm). The values under the Whittle index policy, Policy 1, and Policy 2 are approximated by Monte Carlo simulation with $1,000,000$ samples. We study the following seven cases with various values of N; the results are given in Table 4:

case 1: N arms with $\theta(1) = 0.5$, $\theta(0) = 1.0$, $C = 1.0$,
case 2: N arms with $\theta(1) = 0.5$, $\theta(0) = 0.5$, $C = 1.0$,
case 3: N arms with $\theta(1) = 0.5$, $\theta(0) = 1.0$, $C = 0.5$,
case 4: N arms with $\theta(1) = 0.5$, $\theta(0) = 1.0$, $C = 2.0$,
case 5: N arms with $\theta(1) = 0.9$, $\theta(0) = 1.0$, $C = 1.0$,
case 6: N/2 arms with $\theta(1) = 0.5$, $\theta(0) = 1.0$, $C = 2.0$; N/2 arms with $\theta(1) = 0.5$, $\theta(0) = 0.5$, $C = 1.0$,

N	M	Whittle (95% CI)	Policy 1 (95% CI)	Policy 2 (95% CI)	lower/upper bounds
Case 1:					
2	1	1.4681 (1.4648, 1.4714)	1.1957 (1.1935, 1.1979)	1.2234 (1.2193, 1.2275)	(1.5154992,1.517639)
4	1	1.9119 (1.9084, 1.9154)	1.5450 (1.5426, 1.5474)	1.8741 (1.8706, 1.8776)	
6	1	2.1548 (2.1515, 2.1581)	1.7543 (1.7519, 1.7567)	2.1512 (2.1479, 2.1545)	
8	1	2.3198 (2.3165, 2.3231)	1.9020 (1.8996, 1.9044)	2.3179 (2.3146, 2.3212)	
10	1	2.4412 (2.4379, 2.4445)	2.0125 (2.0101, 2.0149)	2.4381 (2.4348, 2.4414)	
Case 2:					
2	1	1.1374 (1.1354, 1.1394)	1.1968 (1.1946, 1.199)	-0.8995 (−0.9038, −0.8952)	(1.1976295,1.1976295))
4	1	1.4645 (1.4623, 1.4667)	1.5457 (1.5435, 1.5479)	-0.1769 (−0.1810, −0.1728)	
6	1	1.6653 (1.6631, 1.6675)	1.7555 (1.7531, 1.7579)	0.1800 (0.1761, 0.1839)	
8	1	1.8037 (1.8013, 1.8061)	1.9034 (1.901, 1.9058)	0.4143 (0.4104, 0.4182)	
10	1	1.9111 (1.9087, 1.9135)	2.0134 (2.011, 2.0158)	0.5867 (0.5828, 0.5906)	
Case 3:					
2	1	2.6785 (2.6746, 2.6824)	1.1976 (1.1954, 1.1998)	2.6789 (2.675, 2.6828)	(2.690795,2.6942022)
4	1	3.2014 (3.1981, 3.2047)	1.5467 (1.5443, 1.5491)	3.2067 (3.2034, 3.2100)	
6	1	3.4055 (3.4024, 3.4086)	1.7559 (1.7535, 1.7583)	3.4119 (3.4088, 3.415)	
8	1	3.5232 (3.5201, 3.5263)	1.9028 (1.9004, 1.9052)	3.5254 (3.5223, 3.5285)	
10	1	3.6048 (3.6017, 3.6079)	2.0158 (2.0134, 2.0182)	3.6103 (3.6072, 3.6134)	
Case 4:					
2	1	1.1945 (1.1923, 1.1967)	1.1978 (1.1956, 1.2000)	-1.6795 (−1.6842, −1.6748)	(1.1976295,1.1976295)
4	1	1.5420 (1.5398, 1.5442)	1.5468 (1.5444, 1.5492)	-0.7870 (−0.7913, −0.7827)	
6	1	1.7514 (1.7490, 1.7538)	1.7551 (1.7527, 1.7575)	-0.3720 (−0.3761, −0.3679)	
8	1	1.8967 (1.8943, 1.8991)	1.9029 (1.9005, 1.9053)	-0.1089 (−0.1128, −0.105)	
10	1	2.0069 (2.0045, 2.0093)	2.0152 (2.0128, 2.0176)	0.0919 (0.0880, 0.0958)	
Case 5:					
2	1	4.8036 (4.7985, 4.8087)	3.6948 (3.6903, 3.6993)	4.7785 (4.7736, 4.77834)	(4.8408074,4.8647000)
4	1	5.5054 (5.5013, 5.5095)	4.6833 (4.6792, 4.6874)	5.4725 (5.4682, 5.4768)	
6	1	5.7744 (5.7707, 5.7781)	5.1567 (5.1528, 5.1606)	5.7546 (5.7507, 5.7585)	
8	1	5.9097 (5.9062, 5.9132)	5.4304 (5.4267, 5.4341)	5.9160 (5.9123, 5.9197)	
10	1	5.9956 (5.9923, 5.9989)	5.6092 (5.6057, 5.6127)	6.0191 (6.0156, 6.0226)	
Case 6:					
2	1	1.1636 (1.1616, 1.1656)	1.1974 (1.1952, 1.1996)	-1.2804 (−1.2849, −1.2759)	(1.1976295,1.1976295)
4	1	1.5017 (1.4995, 1.5039)	1.5455 (1.5433, 1.5477)	-0.4783 (−0.4824, −0.4742)	
6	1	1.7071 (1.7047, 1.7095)	1.7563 (1.7539, 1.7587)	-0.1007 (−0.1048, −0.0966)	
8	1	1.8487 (1.8463, 1.8511)	1.9033 (1.9009, 1.9057)	0.1426 (0.1387, 0.1465)	
10	1	1.9605 (1.9581, 1.9629)	2.0142 (2.0118, 2.0166)	0.3258 (0.3219, 0.3297)	
Case 7:					
2	1	4.0397 (4.0344, 4.045)	2.7604 (2.7561, 2.7647)	4.0183 (4.0134, 4.0232)	(4.0711303,4.0931025)
4	1	4.8918 (4.8871, 4.8965)	3.7276 (3.7233, 3.7319)	4.7847 (4.7802, 4.7892)	
6	1	5.2900 (5.2857, 5.2943)	4.2742 (4.2701, 4.2783)	5.1406 (5.1365, 5.1447)	
8	1	5.5199 (5.5158, 5.524)	4.6326 (4.6287, 4.6365)	5.3666 (5.3627, 5.3705)	
10	1	5.6692 (5.6653, 5.6731)	4.8892 (4.8853, 4.8931)	5.5210 (5.5171, 5.5249)	

TABLE 4. Results for Problem 2

case 7: N/2 arms with $\theta(1) = 0.9$, $\theta(0) = 1.0$, $C = 1.0$; N/2 arms with $\theta(1) = 0.5$, $\theta(0) = 1.0$, $C = 0.5$.

As we expected, the Whittle index policy resolves the trade-off between repairing arms and ignoring them very effectively. As Policy 1 and Policy 2 behave oppositely when all the arms are unavailable, it is natural that Policy 1 works well when Policy 2 does not, and vice versa. On the other hand, the results show that the Whittle index policy always works at least as good as Policies 1 and 2, which implies that the Whittle indices are effective in deciding whether or not arms should be repaired.

Note that a policy can attain a negative value that is bounded from below by $-C^{max}/(1-\gamma) = -10C^{max}$. Considering this fact, even in comparison to the optimal value, the Whittle index policy performs well in Problem 2 as it does in Problem 1.

10. Conclusion

In this paper, we have studied an important extension of the classical multi-armed bandit problem, in which arms may become unavailable: arms may be intermittently unavailable or they may break down and repair is an option at some cost. Passive arms that may break down and can never get back into the system and the multi-armed bandit problems with switching costs can be handled in this formulation.

Our results are both positive and negative. We showed that problems with availability constraints do not admit an optimal index policy. However, the Whittle index policies we calculated perform in numerical examples very well. Moreover, the index policies for each problem enjoys the following properties:

(i) no index policy performs better uniformly,
(ii) it is optimal for the classical bandit problems and the W-subsidy problems,
(iii) it converges to the Gittins index as the probability of availability approaches to one and to the immediate reward as it approaches to zero.

The Whittle indices can be computed by the value iteration algorithm using a variant of the restart-in problem. Finally, the numerical results are consistent with the near-optimality of Whittle index policies.

Acknowledgment

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Appendix A. Proofs

A.1. **Proof of Proposition 4.1.** Consider the W-subsidy problem with fixed $W \in \mathbb{R}$. The main argument in this proof is that if the process (X, Y) enters a state $(x, 1)$ for some $x \in \mathcal{X}$ (and the arm is available) and if it is optimal to take the passive action, then it must be optimal to take the passive action at every stage after that.

Consider (X, Y) with initial state $(X(0), Y(0)) = (x, 1)$. Let us suppose $(x, 1) \in \Pi(W)$, meaning that it is optimal to take the passive action if the arm is in state $(x, 1)$. Note that X does not change under the passive action, and that when it becomes unavailable (when it enters $(x, 0)$) it is also optimal to make it passive by Condition 4.1. Consequently, when the arm becomes available, the new state is again $(x, 1)$. As we are assuming that $(x, 1) \in \Pi(W)$, the passive action is optimal again. Consequently, once the process (X, Y) enters the state $(x, 1) \in \Pi(W)$, the arm must stay passive forever.

Therefore, the W-subsidy problem is reduced to an optimal stopping problem; finding the best time when the arm is available to switch to the passive action. We restrict our attention to $(\mathcal{F}_t)_{t\geq 0}$-stopping times τ such that $Y(\tau) = 1$ almost surely, and consider the following strategy: until time $\tau - 1$, activate the arm if it is available and choose the passive action otherwise; at and after time τ always choose the passive action. The expected total discounted reward of this strategy is

$$\mathbb{E}_{x,1}^{1,0}\left[\sum_{t=0}^{\tau-1}\gamma^t\left(R^{Y(t)}(X(t),Y(t)) + W1_{\{Y(t)=0\}}\right) + \sum_{t=\tau}^{\infty}\gamma^t W + \gamma^\tau\rho(X(\tau),Y(\tau))\right].$$

On the other hand, stopping immediately in state $(x, 1)$ gives $\sum_{t=0}^{\infty}\gamma^t W + \rho(x, 1) = W/(1 - \gamma) + \rho(x, 1)$. Therefore, $(x, 1) \in \Pi(W)$ if and only if, for every $\tau \in \overline{\mathfrak{S}}$ we have

$$\frac{W}{1-\gamma} + \rho(x,1) \geq \mathbb{E}_{x,1}^{1,0}\left[\sum_{t=0}^{\tau-1}\gamma^t\left(R^{Y(t)}(X(t),Y(t)) + W1_{\{Y(t)=0\}}\right) + \gamma^\tau\rho(X(\tau),Y(\tau)) + \sum_{t=\tau}^{\infty}\gamma^t W\right]. \tag{26}$$

Moreover, the expectation on the right-hand side equals

$$\mathbb{E}_{x,1}^{1,0}\left[\sum_{t=0}^{\tau-1}\gamma^t R^{Y(t)}(X(t),Y(t)) + \gamma^\tau\rho(X(\tau),Y(\tau))\right] + W\mathbb{E}_{x,1}^{1,0}\left[\sum_{t=1}^{\tau-1}\gamma^t 1_{\{Y(t)=0\}} + \frac{\gamma^\tau}{1-\gamma}\right]. \tag{27}$$

Substituting (27) into (26) and some algebra give

$$W \geq (1-\gamma)\frac{\mathbb{E}_{x,1}^{1,0}\left[\sum_{t=0}^{\tau-1}\gamma^t R^{Y(t)}(X(t),Y(t)) + \gamma^\tau\rho(X(\tau),Y(\tau))\right] - \rho(x,1)}{1 - \mathbb{E}_{x,1}^{1,0}\left[(1-\gamma)\sum_{t=1}^{\tau-1}\gamma^t 1_{\{Y(t)=0\}} + \gamma^\tau\right]},$$

for every $\tau \in \overline{\mathfrak{S}}$. Thus, $(x, 1) \in \Pi(W)$ if and only if (7) holds.

A.2. **Proof of Proposition 4.2.** Firstly, the Whittle index when the arm is not available follows from its definition and Condition 4.1. Now consider the case when the arm is available. By Proposition 4.1, $(x,1)\in\Pi(W)$ if and only if (7) is satisfied. Therefore,

$$W_1>W_2 \quad\Longrightarrow\quad \{(x,1);(x,1)\in\Pi(W_1)\}\supseteq\{(x,1);(x,1)\in\Pi(W_2)\},$$

and by Condition 4.1 $W_1>W_2\Longrightarrow\Pi(W_1)\supseteq\Pi(W_2)$; therefore, the arm is indexable, and $W(x,1)\equiv\inf\{W:(x,1)\in\Pi(W)\}$ coincides with the Whittle index.

A.3. **Proof of Corollary 4.1.** If $Y(0)=1$, then by Condition 4.2 $\mathbb{P}^{1,0}$-a.s $Y(t)=1$, $t\geq 0$. Thus, $\overline{\mathfrak{S}}$ can be replaced by $\mathfrak{S}$. Substituting $1_{\{Y(t)=0\}}=0$ into (8) completes the proof.

A.4. **Proof of Corollary 4.3.** We obtain (11) after substituting into (10) the expression

$$\begin{aligned}(28)\quad &\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=1}^{\tau-1}\gamma^t 1_{\{Y(t)=0\}}\right]=\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=1}^{\infty}\gamma^t 1_{\{Y(t)=0\}}1_{\{t<\tau\}}\right]=\sum_{t=1}^{\infty}\mathbb{E}^{1,0}_{x,1}\left[\gamma^t 1_{\{Y(t)=0\}}1_{\{t<\tau\}}\right]\\ &=\sum_{t=1}^{\infty}\gamma^t\mathbb{E}^{1,0}_{x,1}\left[1_{\{Y(t)=0\}}\right]\mathbb{E}^{1,0}_{x,1}\left[1_{\{t<\tau\}}\right]=\sum_{t=1}^{\infty}\gamma^t(1-\theta)\mathbb{E}^{1,0}_{x,1}\left[1_{\{t<\tau\}}\right]=\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=1}^{\tau-1}(1-\theta)\gamma^t\right].\end{aligned}$$

A.5. **Proof of Corollary 4.4.** If $Y(0)=1$, then $Y(t)=1$, $\mathbb{P}^{1,0}$-almost surely, for every $t\geq 0$ by (12). Therefore, $\overline{\mathfrak{S}}$ is the same as $\mathfrak{S}$, and after plugging $\theta\equiv 1$ in (11), we obtain

$$W(x,1)=(1-\gamma)\sup_{\tau\in\mathfrak{S}}\frac{\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right]}{1-\mathbb{E}^{1,0}_{x,1}\left[\gamma^\tau\right]}=\sup_{\tau\in\mathfrak{S}}\frac{\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right]}{\mathbb{E}^{1,0}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t\right]}.$$

A.6. **Proof of Proposition 4.3.** Only in this proof, in order to emphasize the dependence of W and $\mathbb{P}^{1,0}$ on $\theta\in[0,1]$, we replace them with W_θ and $\mathbb{P}^\theta$, respectively. Therefore,

$$W_\theta(x,y)=\begin{cases}(1-\gamma)\sup\limits_{\tau\in\overline{\mathfrak{S}}}\Gamma(\theta,\tau,x)\triangleq\dfrac{\mathbb{E}^{\theta}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\gamma(1-\theta)-\theta\mathbb{E}^{\theta}_{x,1}\left[\gamma^\tau\right]}, & \text{if } y=1,\\ -\infty, & \text{otherwise.}\end{cases}$$

The Gittins index corresponds to $M(x)=W_1(x,1)\equiv(1-\gamma)\sup_{\tau\in\overline{\mathfrak{S}}}\Gamma(1,\tau,x)$. Let $\bar{R}$ be a finite constant such that $|R(x,1)|<\bar{R}$ for every $x\in\mathcal{X}$.

We first prove the convergence to the immediate reward as $\theta\searrow 0$. As immediate stopping attains $R^1(x,1)$, we have $W_\theta(x,1)\geq R(x,1)$. Let

$$\tilde{W}_\theta(x,1)\triangleq\sup_{\tau\in\overline{\mathfrak{S}}}\mathbb{E}^{\theta}_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right].$$

Because $(1-\gamma)\leq 1-\gamma(1-\theta)-\theta\mathbb{E}^{1,0}_{x,1}\left[\gamma^\tau\right]$ for every $\tau\in\overline{\mathfrak{S}}$, we have $\tilde{W}_\theta(x,1)\geq W_\theta(x,1)$. Therefore, it is sufficient to show that $\tilde{W}(x,1)-R^1(x,1)\to 0$ as $\theta\to 0$. Let K be the

next time the arm is available. Then $\mathbb{P}^\theta\{K=k\} = (1-\theta)^{k-1}\theta,\ k \geq 1$ and $\mathbb{E}^\theta\left[\gamma^K\right] = \theta\gamma/(1-\gamma(1-\theta)) \leq \theta\gamma/(1-\gamma)$; therefore,

$$\tilde{W}(x,1) \leq R^1(x,1) + \mathbb{E}^{1,0}_{x,1}\left[\gamma^K \sum_{t=0}^{\infty} \gamma^t \bar{R}\right] < R^1(x,1) + \theta\frac{\bar{R}\gamma}{(1-\gamma)^2}$$

and $R^1(x,1) \leq W_\theta(x,1) \leq \tilde{W}_\theta(x,1) < R^1(x,1) + \theta\bar{R}\gamma/(1-\gamma)^2$ holds for every $x \in \mathcal{X}$, and $W_\theta(x,1)$ converges to the immediate reward $R^1(x,1)$ as $\theta \searrow 0$ uniformly over $x \in \mathcal{X}$. To show the convergence to the Gittins index as $\theta \nearrow 1$, we will need the following lemma.

Lemma A.1. *There is a finite constant B such that $\sup_{\tau\in\overline{\mathfrak{S}},\, x\in\mathcal{X}} |\Gamma(1,\tau,x) - \Gamma(\theta,\tau,x)| < B(1-\theta)$.*

Proof. For $\theta \in (0,1)$, $\tau \in \overline{\mathfrak{S}}$, and $x \in \mathcal{X}$, the difference $|\Gamma(1,\tau,x) - \Gamma(\theta,\tau,x)|$ equals

$$\begin{aligned}
&\left|\frac{\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right]}{1-\mathbb{E}^1_{x,1}\left[\gamma^\tau\right]} - \frac{\mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}\left[\gamma^\tau\right]}\right| \\
&\leq \left|\frac{\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right]}{1-\mathbb{E}^1_{x,1}\left[\gamma^\tau\right]} - \frac{\mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\mathbb{E}^1_{x,1}\left[\gamma^\tau\right]}\right| \\
&\quad + \left|\frac{\mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\mathbb{E}^1_{x,1}\left[\gamma^\tau\right]} - \frac{\mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}\left[\gamma^\tau\right]}\right| \\
&= \frac{\left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]\right|}{1-\mathbb{E}^1_{x,1}\left[\gamma^\tau\right]} \\
&\quad + \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]\left|\frac{1}{1-\mathbb{E}^1_{x,1}\left[\gamma^\tau\right]} - \frac{1}{1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}\left[\gamma^\tau\right]}\right| \\
&\leq \frac{\left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]\right|}{1-\gamma} \\
&\quad + \frac{\bar{R}}{1-\gamma}\left|\frac{1}{1-\mathbb{E}^1_{x,1}\left[\gamma^\tau\right]} - \frac{1}{1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}\left[\gamma^\tau\right]}\right|.
\end{aligned}$$

Now it is sufficient to prove that there exist some finite numbers B_1 and B_2 such that

(i) $\left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]\right| \leq B_1(1-\theta)$,
(ii) $\left|\frac{1}{1-\mathbb{E}^1_{x,1}[\gamma^\tau]} - \frac{1}{1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}[\gamma^\tau]}\right| \leq B_2(1-\theta)$.

To this end, let L be the first time the arm is unavailable. Note that for every $l \geq 1$, the joint conditional $\mathbb{P}^\theta$-distribution of $\{(X(t),Y(t)); 0 \leq t \leq l-1\}$ given $L = l$ is the same as the joint unconditional $\mathbb{P}^1$-distribution of $\{(X(t),Y(t)); 0 \leq t \leq l-1\}$, and we have $\mathbb{P}^\theta\{L=l\} = \theta^{l-1}(1-\theta),\ l \geq 1$ and $\mathbb{E}^\theta_{x,1}\left[\gamma^L\right] = \gamma(1-\theta)/(1-\gamma\theta) < (1-\theta)\gamma/(1-\gamma)$. The

inequality in (i) holds with $B_1 = (1-\theta)2\gamma\bar{R}/(1-\gamma)^2$ because

$$\begin{aligned}
&\left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]\right| \\
&\le \sum_{l=1}^{\infty}\mathbb{P}^\theta_{x,1}\{L=l\}\left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\middle| L=l\right]\right|,
\end{aligned}$$

where for every $l \ge 1$, the absolute difference can be rewritten as

$$\begin{aligned}
&\left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{\tau\le l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}1_{\{\tau\le l-1\}}\middle| L=l\right]\right. \\
&\qquad \left. + \mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{\tau>l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}1_{\{\tau>l-1\}}\middle| L=l\right]\right| \\
&= \left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{\tau>l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}1_{\{\tau>l-1\}}\middle| L=l\right]\right| \\
&\le \left|\mathbb{E}^1_{x,1}\left[\sum_{t=0}^{l-1}\gamma^t R^1(X(t),Y(t))1_{\{\tau>l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=0}^{l-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}1_{\{\tau>l-1\}}\middle| L=l\right]\right| \\
&\qquad + \left|\mathbb{E}^1_{x,1}\left[\sum_{t=l}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{\tau>l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=l}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}1_{\{\tau>l-1\}}\middle| L=l\right]\right| \\
&= \left|\mathbb{E}^1_{x,1}\left[\sum_{t=l}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{\tau>l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\sum_{t=l}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}1_{\{\tau>l-1\}}\middle| L=l\right]\right|
\end{aligned}$$

which is less than $\sum_{t=l}^{\infty}\gamma^t 2\bar{R}$; therefore, the left-hand side of (i) is less than or equal to

$$\sum_{l=1}^{\infty}\mathbb{P}^\theta_{x,1}\{L=l\}\sum_{t=l}^{\infty}\gamma^t 2\bar{R} = \frac{2\bar{R}}{1-\gamma}\mathbb{E}^\theta_{x,1}\left[\gamma^L\right] = (1-\theta)\frac{2\gamma\bar{R}}{(1-\gamma)^2}.$$

The inequality in (ii) holds with $B_2 = \gamma(1-\theta)/(1-\gamma)^2$ because

$$\begin{aligned}
&\left|\frac{1}{1-\mathbb{E}^1_{x,1}[\gamma^\tau]} - \frac{1}{1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}[\gamma^\tau]}\right| = \left|\frac{-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}[\gamma^\tau]+\mathbb{E}^1_{x,1}[\gamma^\tau]}{\left(1-\mathbb{E}^1_{x,1}[\gamma^\tau]\right)\left(1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}[\gamma^\tau]\right)}\right| \\
&\le \frac{\gamma(1-\theta)+\theta\left|\mathbb{E}^\theta_{x,1}[\gamma^\tau]-\mathbb{E}^1_{x,1}[\gamma^\tau]\right|+(1-\theta)\mathbb{E}^1_{x,1}[\gamma^\tau]}{\left(1-\mathbb{E}^1_{x,1}[\gamma^\tau]\right)\left(1-\gamma(1-\theta)-\theta\mathbb{E}^\theta_{x,1}[\gamma^\tau]\right)} \le \frac{(\gamma+1)(1-\theta)+\left|\mathbb{E}^\theta_{x,1}[\gamma^\tau]-\mathbb{E}^1_{x,1}[\gamma^\tau]\right|}{(1-\gamma)^2},
\end{aligned}$$

and $\left|\mathbb{E}^1_{x,1}[\gamma^\tau] - \mathbb{E}^\theta_{x,1}[\gamma^\tau]\right| \le \sum_{n=1}^\infty \mathbb{P}^\theta_{x,1}\{L=l\}\left|\mathbb{E}^1_{x,1}[\gamma^\tau] - \mathbb{E}^\theta_{x,1}[\gamma^\tau|L=l]\right|$ equals

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}^\theta_{x,1}\{L=l\}\Big|\mathbb{E}^1_{x,1}\left[\gamma^\tau 1_{\{\tau\le l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\gamma^\tau 1_{\{\tau\le l-1\}}|L=l\right] \\
+\mathbb{E}^1_{x,1}\left[\gamma^\tau 1_{\{\tau>l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\gamma^\tau 1_{\{\tau>l-1\}}|L=l\right]\Big| \\
= \sum_{n=1}^{\infty} \mathbb{P}^\theta_{x,1}\{L=l\}\Big|\mathbb{E}^1_{x,1}\left[\gamma^\tau 1_{\{\tau>l-1\}}\right] - \mathbb{E}^\theta_{x,1}\left[\gamma^\tau 1_{\{\tau>l-1\}}|L=l\right]\Big| \\
= \sum_{n=1}^{\infty} \mathbb{P}^\theta_{x,1}\{L=l\}\sum_{t=l}^{\infty}\gamma^t\left|\mathbb{P}^1_{x,1}\{\tau=t\} - \mathbb{P}^\theta_{x,1}\{\tau=t|L=l\}\right| \\
\le \sum_{n=1}^{\infty} \mathbb{P}^\theta_{x,1}\{L=l\}\sum_{t=l}^{\infty}\gamma^t = \frac{1}{1-\gamma}\mathbb{E}^\theta_{x,1}\left[\gamma^L\right] < (1-\theta)\frac{\gamma}{(1-\gamma)^2}.
\end{aligned}$$

Therefore, the conclusion of Lemma A.1 follows if $B \triangleq B_1/(1-\gamma) + B_2\bar{R}/(1-\gamma)$. □

Fix $x \in \mathcal{X}$. Let $\tau^1(x)$ and $\tau^\theta(x)$ attain $M(x) = \sup_{\tau\in\overline{\mathfrak{S}}}\ \Gamma(1,\tau,x)$ and $W_\theta(x,1) = \sup_{\tau\in\overline{\mathfrak{S}}}\Gamma(\theta,\tau,x)$, respectively. Then Lemma A.1 implies that

$$\begin{aligned}
\Gamma(1,\tau^1(x),x) \ge \Gamma(1,\tau^\theta(x),x) \ge \Gamma(\theta,\tau^\theta(x),x) - B(1-\theta) \\
\ge \Gamma(\theta,\tau^1(x),x) - B(1-\theta) \ge \Gamma(1,\tau^1(x),x) - 2B(1-\theta),
\end{aligned}$$

which completes the proof of Proposition 4.3 because

$$\sup_{x\in\mathcal{X}}|M(x) - W_\theta(x,1)| \;=\; \sup_{x\in\mathcal{X}}\left|\Gamma(1,\tau^1(x),x) - \Gamma(\theta,\tau^\theta(x),x)\right| \;\le\; 2B(1-\theta) \;\xrightarrow{\theta\uparrow 1}\; 0.$$

A.7. **Proof of Proposition 4.4.** The W-subsidy problem is a special case of Problem 1. To see this, consider the situation where there are only two arms; arm 1 follows a generic stochastic process $(X(t),Y(t))$ as defined in Section 2, and arm 2 is always available and gives a constant reward a. Let $(x_1,1)$ and $(x_2,0)$ denote the current state of arm 1 and arm 2, respectively. Therefore, by Proposition 4.1 it is optimal to rest the arm if and only if $a \ge W(x_1,1)$. That is, the index of arm 1 must be a strict monotone transformation of $W(x_1,1)$ and that of arm 2 must be a monotone transformation of a. The process $(X(t),Y(t))$ of arm 1 is general, and any arm in the problem class has to satisfy the above mentioned property. Moreover,

$$W(x_2,1) = (1-\gamma)\sup_{\tau\in\overline{\mathfrak{S}}}\frac{\mathbb{E}^{1,0}_{x_2,1}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\mathbb{E}^{1,0}_{x_2,1}\left[\sum_{t=1}^{\tau-1}(1-\gamma)\gamma^t 1_{\{Y(t)=0\}} + \gamma^\tau\right]} = \sup_{\tau\in\mathfrak{S}}\frac{\mathbb{E}^{1,0}_{x_2,1}\left[\sum_{t=0}^{\tau-1}\gamma^t a\right]}{\mathbb{E}^{1,0}_{x_2,1}\left[\sum_{t=0}^{\tau-1}\gamma^t\right]} = a.$$

Since any optimal index $\hat{W}(\cdot,\cdot)$ must also be optimal on every W-subsidy problem, it is immediate that any such index must be a strict monotone transformation of $W(\cdot,\cdot)$.

A.8. **Proof of Proposition 4.5.** We give a counterexample in which an arm is available now, but its future availability is highly unlikely: consider the case with two arms where arm 1 is always available, and arm 2 is available with probability $\epsilon \in (0,1)$. Suppose that passive arms do not give rewards and $M = 1$.

The reward from arm 1 changes deterministically under the active action

$$1 \to 100 \to 10 \to 10 \to \ldots \to 10 \to \ldots.$$

Let the corresponding states be $x_{11}, x_{12}, x_{13}, \ldots$. On the other hand, arm 2 does not change its state and it gives a constant reward 40 whenever it is available. Denote its state by x_2. Suppose both arm 1 and arm 2 are currently available, and their states are x_{11} and x_2, respectively. Let $\epsilon = 0.01$ and $\gamma = 0.7$. After obvious choices of stopping times τ in (11), the index for arm 1 satisfies the bounds

$$W_1(x_{11},1) \geq \frac{1+\gamma 100}{1+\gamma} = 41.76,\ W_1(x_{12},1) \geq \frac{100}{1} = 100,\ W_1(x_{1n},1) = \ 10,\ n \geq 3, \tag{29}$$

and the indices for arm 2 must satisfy $W_2(x_2,1) = 40$ and $W_2(x_2,0) = -\infty$.

For contradiction, assume that the index policy defined by (11) is optimal. Then arm 1 must be pulled, if $X_1 = x_{11}$; arm 1 must be pulled, if $X_1 = x_{12}$; arm 2 must be pulled whenever it is available and arm 1 must be pulled otherwise, if $X_1 = x_{13}, \ldots$. That is, the optimal policy is to pull arm 1 twice (at $X_1 = x_{11}$ and x_{12}) and then pull arm 2 whenever it is available and arm 1 otherwise. Therefore, the value function must satisfy

$$\begin{aligned} V((x_{11},1),(x_2,1)) &= 1 + \gamma 100 + \gamma^2\{10(1-\epsilon)+40\epsilon\} + \gamma^3\{10(1-\epsilon)+40\epsilon\} + \ldots. \\ &= 1 + \gamma 100 + \gamma^2 \sum_{k=0}^{\infty} \left\{\gamma^k (10 + 30\epsilon)\right\} \approx 87.8233, \end{aligned}$$

and $V((x_{11},1),(x_2,0)) = V((x_{11},1),(x_2,1))$; however, pulling arm 2 at $((x_{11},1),(x_2,1))$ gives

$$40 + \gamma\{\epsilon V((x_{11},1),(x_2,1)) + (1-\epsilon)V((x_{11},1),(x_2,0))\} \approx 101.4763 > V((x_{11},1),(x_2,1)).$$

This contradicts with the optimality of V. Therefore, the index policy given by (11) cannot be optimal, and by Proposition 4.4 no optimal index policy exists for Problem 1 in general.

A.9. **Proof of Proposition 5.1.** We prove the indexability and obtain the Whittle index of the arm under Condition 5.1. We consider the cases $W < -C(x)$ and $W \geq -C(x)$ separately after the following two lemmas.

Lemma A.2. *For every $x \in \mathcal{X}$ and $W \in \mathbb{R}$, if $(x,0) \in \Pi(W)$, then $V((x,0),W) = W/(1-\gamma)$, which is obtained by taking the passive action all the time.*

Proof. Both X and Y do not change under the passive action. If $(x,0) \in \Pi(W)$, then the passive action remains optimal forever. Consequently, the expected total discounted reward starting in $(x,0)$ becomes $V((x,0),W) = \sum_{t=0}^{\infty} \gamma^t W = W/(1-\gamma)$. □

Lemma A.3. *For every $x \in \mathcal{X}$ and $W \in \mathbb{R}$, if $(x,1) \in \Pi(W)$, then the stochastic process (X,Y) starting in $(x,1)$ visits only $(x,1)$ and/or $(x,0)$ under the optimal policy, and*

$$V((x,1),W) = \max\left\{ \mathbb{E}^{0,1}_{x,1}\left[\sum_{t=0}^{\infty} \gamma^t \left\{ W 1_{\{Y(t)=1\}} - C(x) 1_{\{Y(t)=0\}} \right\} \right], \frac{W}{1-\gamma} \right\}, \tag{30}$$

where $\mathbb{P}^{0,1}$ is the probability law induced by the policy that activates the arm as long as it is unavailable and leaves it in rest otherwise.

Proof. First, note that X does not change under the passive action and when the arm is not available. Therefore, the next time the arm is available, $X = x$. As we are assuming that $(x,1) \in \Pi(W)$, the passive action must be taken whenever the arm is available. Therefore, only $(x,0)$ and $(x,1)$ will be visited by the process.

Now the optimal policy must be one of the following two: the policy under which the arm is passive when it is in state $(x,1)$ or in $(x,0)$, and the policy under which the arm is passive when it is in $(x,1)$, but is active when it is in $(x,0)$. The value under the former policy is $W/(1-\gamma)$ by Lemma A.2, and the value under the latter is the expectation in (30). □

Lemma A.4. *If $W < -C(x)$, and Condition 5.2 holds, then $(x,y) \notin \Pi(W)$ for every $y \in \{0,1\}$; therefore, $\Pi(W) = \emptyset$.*

Proof. Suppose that $(x,0) \in \Pi(W)$ for some $x \in \mathcal{X}$. By Lemma A.2,

$$V((x,0),W) = W/(1-\gamma). \tag{31}$$

However, a lower bound on $V((x,0),W)$ can be obtained by considering the policy under which the arm is active at $(x,0)$ and passive otherwise. Because $W < -C(x)$, this policy gives $V((x,0),W) > W/(1-\gamma)$, which contradicts with (31) and implies that

$$(x,0) \notin \Pi(W), \quad x \in \mathcal{X}. \tag{32}$$

Suppose now $(x,1) \in \Pi(W)$ for some $x \in \mathcal{X}$. Then, by Lemma A.3 and (32), we obtain

$$V((x,1),W) = \mathbb{E}^{0,1}_{x,1}\left[\sum_{t=0}^{\infty} \gamma^t \left\{ W 1_{\{Y(t)=1\}} - C(x) 1_{\{Y(t)=0\}} \right\} \right] \leq -\frac{C(x)}{1-\gamma}.$$

However, this contradicts with the lower bound obtained by applying the policy under which the arm is always active; namely,

$$V((x,1),W) = \mathbb{E}^{1,1}_{x,1}\left[\sum_{t=0}^{\infty} \gamma^t \left\{ R^1(X(t),Y(t)) 1_{\{Y(t)=1\}} - C(X(t)) 1_{\{Y(t)=0\}} \right\} \right] > -\frac{C(x)}{1-\gamma},$$

where the last inequality holds under Condition 5.2. Therefore, $(x,1) \notin \Pi(W), x \in \mathcal{X}$. $\square$

Lemma A.5. *If $W \geq -C(x)$, then $(x,y) \in \Pi(W)$ if and only if, for every $\tau \in \mathfrak{S}$,*

$$W \geq (1-\gamma)\frac{\mathbb{E}^{1,1}_{x,y}\left[\sum_{t=0}^{\tau-1}\gamma^t R^1(X(t),Y(t))\right]}{1-\mathbb{E}^{1,1}_{x,y}\left[\gamma^\tau\right]}.$$

Proof. The main argument in the proof is that in the W-subsidy problem, once the passive action is optimal, it remains optimal to take the passive action thereafter. This follows for $y=0$ immediately from Lemma A.2.

Suppose now $(x,1) \in \Pi(W)$ for some $x \in \mathcal{X}$. By Lemma A.3, starting at $(x,1)$ under the optimal policy, only $(x,0)$ and $(x,1)$ will be visited by the process (X,Y), and (30) holds. Moreover, $W \geq -C(x)$ implies

$$\mathbb{E}^{0,1}_{x,1}\left[\sum_{t=0}^{\infty}\gamma^t\left\{W1_{\{Y(t)=1\}} - C(x)1_{\{Y(t)=0\}}\right\}\right] \leq \frac{W}{1-\gamma};$$

thus, $V((x,1),W) = W/(1-\gamma)$, and the optimal policy always chooses the passive action.

Using the argument above, the W-subsidy problem is reduced to an optimal stopping problem as in the proof of Proposition 4.2. The optimal strategy must choose the active action until some stopping time τ and choose the passive action at and after time τ. The main difference with Problem 1 is that it may be optimal to stop even when $Y=0$. If we switch from the active action to the passive action at some stopping time τ, then the expected total discounted reward will be

$$\mathbb{E}^{1,1}_{x,y}\left[\sum_{t=0}^{\tau-1}R^1(X(t),Y(t)) + \sum_{t=\tau}^{\infty}\gamma^t W\right].$$

As in the proof of Proposition 4.2, $(x,y) \in \Pi(W)$ if and only if immediate stopping achieves greater value than the displayed expectation for any positive stopping time $\tau \in \mathfrak{S}$. Since immediate stopping yields $W/(1-\gamma)$, we have $(x,y) \in \Pi(W)$ if and only if

$$\frac{W}{1-\gamma} \geq \mathbb{E}^{1,1}_{x,y}\left[\sum_{t=0}^{\tau-1}R^1(X(t),Y(t)) + \sum_{t=\tau}^{\infty}\gamma^t W\right] \quad \text{for every } \tau \in \mathfrak{S}. \qquad \square$$

Proposition 5.1 follows immediately from Lemmas A.4 and A.5.

A.10. **Proof of Proposition 5.2.** The monotonicity of $W \mapsto \Pi(W)$ is clear by Proposition 5.1; therefore, the arm is indexable. By Proposition 5.1 and by definition of the Whittle index (4), the Whittle index at (x,y) is given by (16).

A.11. **Proof of Proposition 5.4.** Consider a variant of the situation in the proof for Proposition 4.5 (see Proof of Proposition 4.5 in A.8) with the same value of ϵ and γ. Arm 1 is defined the same as in the proof of Proposition 4.5, and arm 2 again gives a constant reward 40 whenever it is available. Arm 2 breaks down with probability $(1-\epsilon)$, and it is repaired with probability 1 the next time when it is repaired. The repair costs $C = 100$. The index for arm 1 still satisfies the inequalities in (29) while the index for arm 2 satisfies $W_2(x_2, 1) \leq 40$, and

$$W_2(x_2,0) = \sup_{\tau\in\mathfrak{S}} \frac{\mathbb{E}^{1,1}_{x_2,0}\left[\sum_{t=0}^{\tau-1}\gamma^t R(X(t),Y(t))\right]}{\mathbb{E}^{1,1}_{x_2,0}\left[\sum_{t=0}^{\tau-1}\gamma^t\right]} \leq \sup_{\tau\in\mathfrak{S}} \mathbb{E}^{1,1}_{x_2,0}\left[\sum_{t=0}^{\tau-1}\gamma^t R(X(t),Y(t))\right]$$
$$\leq -C + \gamma 40 + \gamma^2 40 + \ldots = -C + \frac{40\gamma}{1-\gamma} \leq 10.$$

Therefore, the index policy pulls arm 1 twice (at $X_1 = x_{11}$ and x_{12}) and then arm 2 whenever it is available and arm 1 otherwise. Therefore, the value function satisfies

$$V((x_{11},1),(x_2,1)) = 1 + \gamma 100 + \gamma^2\left\{10(1-\epsilon^2) + 40\epsilon^2\right\} + \gamma^3\left\{10(1-\epsilon^3) + 40\epsilon^3\right\} + \ldots.$$
$$= 1 + \gamma 100 + \gamma^2 \sum_{k=0}^{\infty}\left\{\gamma^k\left(10 + 30\epsilon^k\right)\right\} \approx 102.1370,$$

and $V((x_{11},1),(x_2,0)) = V((x_{11},1),(x_2,1))$; however, pulling arm 2 at $((x_{11},1),(x_2,1))$ gives

$$40 + \gamma\left\{\epsilon V((x_{11},1),(x_2,1)) + (1-\epsilon)V((x_{11},1),(x_2,0))\right\} = 111.4959 > V((x_{11},1),(x_2,1)),$$

which contradicts with the optimality of V. Therefore, the index policy defined by (16) cannot be optimal, and by Proposition 5.3 there is no optimal index policy for Problem 2.

A.12. **Proof of Proposition 6.1.** Since $\nu_{\tilde{x},1} = (1-\gamma)R^1(\tilde{x},1) + \gamma\sum_{x'\in\mathcal{X}} p_{\tilde{x}x'}[\theta^1(\tilde{x},1)\nu_{x',1}$ $+(1-\theta^1(\tilde{x},1))\nu_{x',0}]$, we have

$$\nu_{x,1} = \max\left\{(1-\gamma)R^1(x,1) + \gamma\sum_{x'\in\mathcal{S}} p_{xx'}\left[\theta^1(x,1)\nu_{x',1} + (1-\theta^1(x,1))\nu_{x',0}\right], \nu_{\tilde{x},1}\right\}, \quad x\in\mathcal{X}.$$

Together with (20), it is easy to see that

$$\nu_{\tilde{x},1} = \sup_{\tau\in\overline{\mathfrak{S}}} \mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=0}^{\tau-1}\gamma^t\left((1-\gamma)R^1(X(t),Y(t))1_{\{Y(t)=1\}} + (1-\gamma)\nu_{\tilde{x},1}1_{\{Y(t)=0\}}\right) + \gamma^\tau\nu_{\tilde{x},1}\right],$$

which implies for every $\tau \in \overline{\mathfrak{S}}$ that

$$\nu_{\tilde{x},1} \geq \mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=0}^{\tau-1}\gamma^t\left((1-\gamma)R^1(X(t),Y(t))1_{\{Y(t)=1\}}+(1-\gamma)\nu_{\tilde{x},1}1_{\{Y(t)=0\}}\right)+\gamma^\tau\nu_{\tilde{x},1}\right],$$

$$\nu_{\tilde{x},1}\left\{1-\mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=0}^{\tau-1}(1-\gamma)\gamma^t 1_{\{Y(t)=0\}}+\gamma^\tau\right]\right\} \geq \mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=0}^{\tau-1}\gamma^t(1-\gamma)R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right].$$

Hence,

$$\nu_{\tilde{x},1} \geq \sup_{\tau\in\overline{\mathfrak{S}}}\frac{\mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=0}^{\tau-1}\gamma^t(1-\gamma)R^1(X(t),Y(t))1_{\{Y(t)=1\}}\right]}{1-\mathbb{E}^{1,0}_{\tilde{x},1}\left[\sum_{t=0}^{\tau-1}(1-\gamma)\gamma^t 1_{\{Y(t)=0\}}+\gamma^\tau\right]}. \tag{33}$$

However, we also have $\nu_{\tilde{x},1} = (1-\gamma)R^1(x,1)+\gamma\sum_{x'\in\mathcal{X}}p_{xx'}\left[\theta^1(x,1)\nu_{x',1}+(1-\theta^1(x,1))\nu_{x',0}\right]$, which implies that there exists $\tau^* \in \overline{\mathfrak{S}}$ such that $\tau^* > 0$ a.s. and

$$\nu_{\tilde{x},1} = \mathbb{E}^{1,0}_{\tilde{x},1}\left[\left(\sum_{t=0}^{\tau^*-1}\gamma^t\left((1-\gamma)R^1(X(t),Y(t))1_{\{Y(t)=1\}}+(1-\gamma)\nu_{\tilde{x},1}1_{\{Y(t)=0\}}\right)\right)+\gamma^{\tau^*}\nu_{\tilde{x},1}\right],$$

and the arguments similar to the above show that the equality holds in (33).

Appendix B. Whittle index table for Problem 1

Case 1: $\theta = 0.1$

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.5534	0.3713	0.2766	0.2193	0.1540	0.1181	0.0956	0.0484	0.0238
2	0.7019	0.5320	0.4263	0.3546	0.2643	0.2100	0.1740	0.0929	0.0475
3	0.7740	0.6255	0.5226	0.4483	0.3479	0.2837	0.2392	0.1332	0.0700
4	0.8172	0.6870	0.5909	0.5174	0.4139	0.3443	0.2945	0.1700	0.0914
6	0.8670	0.7634	0.6807	0.6137	0.5117	0.4386	0.3835	0.2346	0.1313
8	0.8951	0.8093	0.7378	0.6773	0.5814	0.5088	0.4522	0.2897	0.1678
10	0.9132	0.8401	0.7772	0.7227	0.6334	0.5633	0.5069	0.3373	0.2013
20	0.9529	0.9108	0.872	0.8035	0.7729	0.7182	0.6706	0.5030	0.3348
40	0.9749	0.9521	0.9303	0.9094	0.8703	0.8520	0.8012	0.6681	0.5010

Case 2: $\theta = 0.3$

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.6135	0.4191	0.3131	0.2477	0.1726	0.1313	0.1054	0.0521	0.0255
2	0.7414	0.5707	0.4594	0.383	0.2851	0.2259	0.1865	0.0983	0.0497
3	0.8023	0.6564	0.5513	0.4737	0.3682	0.3	0.2525	0.1395	0.0727
4	0.8388	0.7123	0.6156	0.5403	0.433	0.3602	0.3078	0.1768	0.0944
6	0.881	0.7814	0.6995	0.632	0.528	0.453	0.3962	0.2481	0.1348
8	0.9051	0.8229	0.7524	0.6922	0.5955	0.5215	0.4637	0.297	0.1716
10	0.9208	0.8508	0.7892	0.7352	0.6456	0.5747	0.5174	0.3444	0.2053
20	0.9561	0.9155	0.8778	0.8427	0.7798	0.7252	0.6775	0.5087	0.3388
40	0.9763	0.9542	0.9328	0.9123	0.8737	0.838	0.8051	0.672	0.5042

Case 3: $\theta = 0.5$

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.6496	0.4502	0.3378	0.2676	0.1861	0.1412	0.113	0.0551	0.0266
2	0.7649	0.5954	0.4811	0.4021	0.2996	0.2372	0.1955	0.1023	0.0514
3	0.8194	0.6758	0.5702	0.4904	0.3821	0.3112	0.2618	0.1441	0.0746
4	0.852	0.7284	0.6316	0.5555	0.4458	0.3711	0.3171	0.1816	0.0966
6	0.8899	0.7929	0.7117	0.6441	0.5391	0.4627	0.4048	0.2469	0.1373
8	0.9117	0.8317	0.7621	0.7021	0.605	0.5302	0.4716	0.302	0.1742
10	0.9259	0.8578	0.7971	0.7434	0.6537	0.5825	0.5246	0.3493	0.2079
20	0.9582	0.9188	0.8816	0.8469	0.7844	0.7299	0.6822	0.5127	0.3414
40	0.9772	0.9555	0.9345	0.9143	0.8759	0.8405	0.8076	0.6746	0.5063

Case 4: $\theta = 0.7$

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.6751	0.4736	0.3568	0.2834	0.1971	0.1492	0.1192	0.0576	0.0275
2	0.7817	0.6138	0.4975	0.4168	0.3109	0.2461	0.2028	0.1056	0.0527
3	0.8317	0.6905	0.5845	0.5032	0.393	0.3201	0.2694	0.1478	0.0762
4	0.8616	0.7403	0.6438	0.5672	0.4559	0.3798	0.3244	0.1855	0.0984
6	0.8965	0.8015	0.721	0.6533	0.5478	0.4704	0.4117	0.2509	0.1393
8	0.9166	0.8384	0.7696	0.7098	0.6124	0.5371	0.4778	0.306	0.1763
10	0.9298	0.8632	0.8032	0.7498	0.6601	0.5887	0.5304	0.3532	0.2101
20	0.9599	0.9213	0.8846	0.8502	0.788	0.7336	0.6859	0.5158	0.3435
40	1 0.978	0.9566	0.9359	0.9158	0.8777	0.8424	0.8096	0.6766	0.508

Case 5: $\theta = 0.9$

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.6946	0.4921	0.3727	0.2964	0.2062	0.1561	0.1245	0.0598	0.0282
2	0.7946	0.6284	0.5106	0.4289	0.3204	0.2536	0.209	0.1085	0.0538
3	0.8412	0.7022	0.596	0.5137	0.4019	0.3276	0.2757	0.151	0.0776
4	0.8691	0.7499	0.6537	0.5768	0.4642	0.387	0.3306	0.1889	0.0999
6	0.9017	0.8085	0.7286	0.661	0.5551	0.4768	0.4176	0.2544	0.141
8	0.9205	0.8439	0.7757	0.7161	0.6187	0.543	0.4832	0.3095	0.1781
10	0.933	0.8677	0.8083	0.7551	0.6656	0.594	0.5354	0.3566	0.2119
20	0.9614	0.9234	0.8872	0.853	0.7911	0.7368	0.689	0.5185	0.3453
40	0.9786	0.9576	0.937	0.9171	0.8792	0.844	0.8113	0.6784	0.5095

Case 6: $\theta = 1.0$

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.703	0.5002	0.3797	0.3022	0.2104	0.1593	0.1271	0.0607	0.0286
2	0.8002	0.6348	0.5165	0.4343	0.3247	0.2571	0.2119	0.1098	0.0543
3	0.8454	0.7073	0.6012	0.5186	0.406	0.331	0.2787	0.1524	0.0782
4	0.8724	0.7541	0.6581	0.5811	0.468	0.3903	0.3336	0.1904	0.1006
6	0.9041	0.8117	0.7321	0.6644	0.5584	0.4799	0.4204	0.2559	0.1417
8	0.9224	0.8464	0.7785	0.7191	0.6216	0.5459	0.4859	0.311	0.1789
10	0.9346	0.8699	0.8107	0.7577	0.6682	0.5966	0.538	0.3581	0.2128
20	0.962	0.9244	0.8883	0.8543	0.7924	0.7382	0.6904	0.5197	0.3461
40	0.9789	0.958	0.9376	0.9177	0.8799	0.8448	0.8121	0.6792	0.5101

Appendix C. Whittle index table for Problem 2

Case 1: $\theta(0) = 0.5, \theta(1) = 0.5, C = 1.0$

y=1:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.5506	0.3694	0.2754	0.2186	0.1539	0.1184	0.096	0.0491	0.0247
2	0.7003	0.5306	0.4253	0.354	0.2641	0.2101	0.1742	0.0935	0.0483
3	0.7732	0.6246	0.5219	0.4477	0.3477	0.2837	0.2394	0.1338	0.0708
4	0.8169	0.6864	0.5904	0.5171	0.4138	0.3444	0.2947	0.1705	0.0922
6	0.8672	0.7633	0.6806	0.6136	0.5118	0.4388	0.3838	0.2352	0.1321
8	0.8955	0.8096	0.7379	0.6775	0.5816	0.509	0.4525	0.2903	0.1686
10	0.9138	0.8405	0.7775	0.723	0.6337	0.5636	0.5073	0.3379	0.2021
20	0.9538	0.9116	0.8728	0.8371	0.7736	0.7188	0.6712	0.5037	0.3356
40	0.9759	0.9531	0.9312	0.9103	0.8712	0.8352	0.8021	0.6689	0.5018

y=0:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	-0.1184	-0.2235	-0.28	-0.3147	-0.3549	-0.3771	-0.3912	-0.4206	-0.4355
2	-0.0446	-0.1388	-0.1993	-0.2407	-0.2936	-0.3256	-0.347	-0.3952	-0.4221
3	-0.0095	-0.0905	-0.1482	-0.1906	-0.248	-0.2851	-0.3109	-0.3726	-0.4094
4	0.0113	-0.0591	-0.1124	-0.1537	-0.2123	-0.2521	-0.2806	-0.3522	-0.3974
6	0.0352	-0.0203	-0.0656	-0.1028	-0.1599	-0.201	-0.2321	-0.3165	-0.3753
8	0.0486	0.003	-0.0361	-0.0693	-0.1226	-0.1632	-0.1949	-0.2863	-0.3551
10	0.0573	0.0185	-0.0156	-0.0455	-0.0949	-0.1338	-0.1654	-0.2603	-0.3366
20	0.0765	0.0543	0.0334	0.014	-0.0207	-0.0508	-0.0771	-0.1699	-0.2633
40	0.0875	0.0753	0.0636	0.0523	0.031	0.0114	-0.0068	-0.0801	-0.1723

Case 2: $\theta(0) = 1.0, \theta(1) = 0.5, C = 0.5$

y=1:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.5506	0.3694	0.2754	0.2186	0.1539	0.1184	0.096	0.0491	0.0247
2	0.7003	0.5306	0.4253	0.354	0.2641	0.2101	0.1742	0.0935	0.0483
3	0.7732	0.6246	0.5219	0.4477	0.3477	0.2837	0.2394	0.1338	0.0708
4	0.8169	0.6864	0.5904	0.5171	0.4138	0.3444	0.2947	0.1705	0.0922
6	0.8672	0.7633	0.6806	0.6136	0.5118	0.4388	0.3838	0.2352	0.1321
8	0.8955	0.8096	0.7379	0.6775	0.5816	0.509	0.4525	0.2903	0.1686
10	0.9138	0.8405	0.7775	0.723	0.6337	0.5636	0.5073	0.3379	0.2021
20	0.9538	0.9116	0.8728	0.8371	0.7736	0.7188	0.6712	0.5037	0.3356
40	0.9759	0.9531	0.9312	0.9103	0.8712	0.8352	0.8021	0.6689	0.5018

y=0:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.2757	0.1415	0.0679	0.0223	-0.0305	-0.0599	-0.0784	-0.1171	-0.1367
2	0.3631	0.2453	0.1681	0.115	0.0468	0.0053	-0.0224	-0.0849	-0.1196
3	0.4044	0.3037	0.231	0.177	0.1038	0.0561	0.0229	-0.0564	-0.1036
4	0.4289	0.3417	0.2748	0.2226	0.1481	0.0975	0.061	-0.0307	-0.0885
6	0.457	0.3885	0.3319	0.2852	0.2132	0.1611	0.1216	0.0141	-0.0606
8	0.4728	0.4165	0.3679	0.3263	0.2593	0.208	0.1679	0.0521	-0.0352
10	0.4831	0.4352	0.3928	0.3555	0.2935	0.2445	0.2047	0.0847	-0.0121
20	0.5059	0.4784	0.4526	0.4284	0.385	0.3473	0.3143	0.1976	0.0799
40	0.5191	0.504	0.4895	0.4754	0.4489	0.4243	0.4015	0.3097	0.1937

Case 3: $\theta(0) = 1.0, \theta(1) = 0.5, C = 1.0$

y=1:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.5506	0.3694	0.2754	0.2186	0.1539	0.1184	0.096	0.0491	0.0247
2	0.7003	0.5306	0.4253	0.354	0.2641	0.2101	0.1742	0.0935	0.0483
3	0.7732	0.6246	0.5219	0.4477	0.3477	0.2837	0.2394	0.1338	0.0708
4	0.8169	0.6864	0.5904	0.5171	0.4138	0.3444	0.2947	0.1705	0.0922
6	0.8672	0.7633	0.6806	0.6136	0.5118	0.4388	0.3838	0.2352	0.1321
8	0.8955	0.8096	0.7379	0.6775	0.5816	0.509	0.4525	0.2903	0.1686
10	0.9138	0.8405	0.7775	0.723	0.6337	0.5636	0.5073	0.3379	0.2021
20	0.9538	0.9116	0.8728	0.8371	0.7736	0.7188	0.6712	0.5037	0.3356
40	0.9759	0.9531	0.9312	0.9103	0.8712	0.8352	0.8021	0.6689	0.5018

Case 3: $\theta(0) = 1.0, \theta(1) = 0.5, C = 1.0$ (continued)

y=0:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.1204	-0.0137	-0.0873	-0.1329	-0.1857	-0.215	-0.2336	-0.2723	-0.2919
2	0.208	0.0901	0.013	-0.0401	-0.1084	-0.1498	-0.1775	-0.2401	-0.2748
3	0.2493	0.1485	0.0758	0.0219	-0.0514	-0.099	-0.1322	-0.2115	-0.2588
4	0.2738	0.1866	0.1196	0.0675	-0.007	-0.0577	-0.0942	-0.1858	-0.2437
6	0.3018	0.2334	0.1768	0.1301	0.0581	0.0059	-0.0336	-0.141	-0.2158
8	0.3176	0.2614	0.2127	0.1711	0.1041	0.0529	0.0128	-0.1031	-0.1904
10	0.3279	0.2801	0.2376	0.2003	0.1383	0.0893	0.0495	-0.0705	-0.1672
20	0.3507	0.3233	0.2974	0.2733	0.2298	0.1921	0.1592	0.0424	-0.0753
40	0.3639	0.3489	0.3343	0.3203	0.2937	0.2691	0.2464	0.1545	0.0386

Case 4: $\theta(0) = 1.0, \theta(1) = 0.5, C = 2.0$

y=1:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.5506	0.3694	0.2754	0.2186	0.1539	0.1184	0.096	0.0491	0.0247
2	0.7003	0.5306	0.4253	0.354	0.2641	0.2101	0.1742	0.0935	0.0483
3	0.7732	0.6246	0.5219	0.4477	0.3477	0.2837	0.2394	0.1338	0.0708
4	0.8169	0.6864	0.5904	0.5171	0.4138	0.3444	0.2947	0.1705	0.0922
6	0.8672	0.7633	0.6806	0.6136	0.5118	0.4388	0.3838	0.2352	0.1321
8	0.8955	0.8096	0.7379	0.6775	0.5816	0.509	0.4525	0.2903	0.1686
10	0.9138	0.8405	0.7775	0.723	0.6337	0.5636	0.5073	0.3379	0.2021
20	0.9538	0.9116	0.8728	0.8371	0.7736	0.7188	0.6712	0.5037	0.3356
40	0.9759	0.9531	0.9312	0.9103	0.8712	0.8352	0.8021	0.6689	0.5018

y=0:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	-0.1899	-0.324	-0.3976	-0.4432	-0.496	-0.5253	-0.5439	-0.5826	-0.6022
2	-0.1023	-0.2202	-0.2973	-0.3504	-0.4187	-0.4601	-0.4878	-0.5504	-0.5851
3	-0.061	-0.1618	-0.2345	-0.2884	-0.3617	-0.4093	-0.4425	-0.5218	-0.5691
4	-0.0365	-0.1237	-0.1907	-0.2428	-0.3173	-0.368	-0.4045	-0.4961	-0.554
6	-0.0085	-0.0769	-0.1335	-0.1802	-0.2522	-0.3044	-0.3439	-0.4513	-0.5261
8	0.0073	-0.0489	-0.0976	-0.1392	-0.2062	-0.2574	-0.2975	-0.4134	-0.5007
10	0.0176	-0.0302	-0.0727	-0.11	-0.172	-0.221	-0.2608	-0.3808	-0.4775
20	0.0404	0.013	-0.0129	-0.037	-0.0805	-0.1182	-0.1511	-0.2679	-0.3856
40	0.0536	0.0386	0.024	0.01	-0.0166	-0.0412	-0.0639	-0.1558	-0.2717

Case 5: $\theta(0) = 1.0, \theta(1) = 0.9, C = 1.0$

y=1:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.6458	0.4484	0.3369	0.2668	0.1853	0.1404	0.1123	0.0549	0.0266
2	0.7625	0.593	0.4791	0.4004	0.2983	0.2362	0.1948	0.1021	0.0514
3	0.8177	0.6738	0.5683	0.4888	0.3808	0.3102	0.261	0.1438	0.0746
4	0.8507	0.7267	0.6301	0.5541	0.4447	0.3702	0.3163	0.1813	0.0966
6	0.8891	0.7918	0.7105	0.6429	0.5382	0.4619	0.4041	0.2466	0.1373
8	0.9111	0.8309	0.7612	0.7012	0.6042	0.5295	0.471	0.3017	0.1742
10	0.9255	0.8572	0.7964	0.7426	0.653	0.5819	0.524	0.3491	0.2079
20	0.9582	0.9186	0.8814	0.8467	0.7841	0.7296	0.6818	0.5125	0.3414
40	0.9773	0.9556	0.9345	0.9142	0.8759	0.8404	0.8075	0.6745	0.5064

y=0:

$a\backslash b$	1	2	3	4	6	8	10	20	40
1	0.438	0.2645	0.1772	0.125	0.0648	0.0314	0.0101	-0.0351	-0.0588
2	0.5704	0.4124	0.3145	0.2485	0.1655	0.1155	0.082	0.0062	-0.0368
3	0.635	0.4977	0.4028	0.3343	0.2419	0.1827	0.1416	0.0432	-0.016
4	0.6739	0.5537	0.4651	0.3976	0.3023	0.2381	0.1922	0.0769	0.0037
6	0.7187	0.6234	0.5472	0.4855	0.3918	0.3244	0.2736	0.1361	0.0404
8	0.7438	0.6651	0.5992	0.5436	0.4554	0.3885	0.3364	0.1866	0.0739
10	0.7598	0.693	0.6351	0.585	0.5028	0.4383	0.3864	0.2301	0.1046
20	0.7946	0.7564	0.721	0.6883	0.6299	0.5796	0.5358	0.3817	0.2269
40	0.8137	0.7931	0.7733	0.7542	0.7184	0.6854	0.6549	0.5326	0.379

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