OPTIMAL STOPPING PROBLEMS FOR ASSET MANAGEMENT

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Abstract. An asset manager invests the savings of some investors in a portfolio of defaultable bonds. The manager pays the investors coupons at a constant rate and receives management fee proportional to the value of portfolio. She also has the right to walk out of the contract at any time with the net terminal value of the portfolio after the payment of investors' initial funds, but is not responsible for any deficit. To control the principal losses, investors may buy from the manager a limited protection which terminates the agreement as soon as the value of the portfolio drops below a predetermined threshold. We assume that the value of the portfolio is a jump-diffusion process and find optimal termination rule of the manager with and without a protection. We also derive the indifference price of a limited protection. We illustrate the solution method on a numerical example. The motivation comes from the collateralized debt obligations.

1. Introduction

We study two optimal stopping problems of an institutional asset manager hired by ordinary investors who do not have access to certain asset classes. The investors entrust their initial funds in the amount of L to the asset manager. As long as the contract is alive, the investors receive coupon payments from the asset manager on their initial funds at a fixed rate (higher than the risk-free interest rate). In return, the asset manager collects dividend or management fee (at a fixed rate on the market value of the portfolio). At any time, the asset manager has the right to terminate the contract and to walk away with the net terminal value of the portfolio after the payment of the investors' initial funds. However, she is not financially responsible for any amount of shortfall. The asset manager's *first problem* is to find a nonanticipative stopping rule which maximizes her expected discounted total income.

Under the original contract, investors face the risk of losing all or some part of their initial funds. Suppose that the asset manager offers the investors a limited protection against this risk, in the form that the new contract will terminate as soon as the market value of the portfolio goes below a predetermined threshold. The asset manager's *second problem* is to find the fair price for the limited protection and the best time to terminate the contract under this additional clause.

We assume that the market value X of the asset manager's portfolio follows a geometric Brownian motion subject to downward jumps which occur according to an independent Poisson process. As explained in detail in the next section, both the problems and the setting are motivated by those faced by the managers responsible for the portfolios of defaultable bonds, for example, as in *collateralized debt obligations* (CDOs). For a detailed description and the valuation of CDOs, we

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refer the reader to Duffie and Gârleanu [16], Goodman and Fabozzi [23], Egami and Esteghamat [18] and Hull and White [19]. Briefly, a CDO is a derivative security on a portfolio of bonds, loans, or other credit risky claims. Cash flows from a collateral portfolio are divided into various quality/yield tranches which are then sold to investors. In our setting, for example, the times of the (downward) jumps in the portfolio value process can be thought as the default times of individual bonds in the portfolio.

The difference between the real-world CDOs and our setting is that a CDO has a pre-determined maturity while we assume an infinite time horizon. However, a typical CDO contract has a term of 10-15 years (much longer than, for example, finite-maturity American-type stock options) and is often extendable with the investors' consent. Hence our perpetuality assumption is a reasonable approximation of the reality. We believe that our analysis is also applicable in certain other financial and real-options settings with no fixed maturity, e.g., open-end mutual funds, outsourcing the maintenance of computing, printing or internet facilities in a company or in a university.

To find the solutions of the asset manager's aforementioned problems, we first model them as optimal stopping problems for a suitable jump-diffusion process under a risk-neutral probability measure. By separating the jumps from the diffusion part by means of a suitable dynamic programming operator, similarly to the approach used by Dayanik, Poor, and Sezer [12] and Dayanik and Sezer [13] for the solutions of sequential statistics problems, we solve the optimal stopping problems by means of successive approximations, which not only lead to accurate and efficient numerical algorithms but also allow us to establish concretely the form of optimal stopping strategies. The idea of stripping the jumps from the diffusion part of a jump-diffusion process was ispired by the seminal work of Davis [8, 9] on piecewise-deterministic Markov processes and the personal discussions of one of the authors with E. Çınlar (see also his talk on the web, Çınlar [4]).

Without any protection, the optimal rule of the asset manager turns out to terminate the contract if the market value of the portfolio X becomes too small or too large; i.e., as soon as X exits an interval (a, b) for some suitable constants $0 < a < b < \infty$.

In the presence of limited protection (provided to the investors by the asset manager for a fee) at some level $\ell \in (a, L]$, it is optimal for the asset manager to terminate the contract as soon as the value X of the portfolio exits an interval (ℓ, m) for some suitable $m \in [\ell, b)$. Namely, if the protection is binding, i.e., $\ell \in (a, b)$, then the asset manager's optimal continuation region shrinks. *In other words, investors can have limited protection only if they are also willing to give up in part from the upside-potential of their managed portfolio.* "Total protection" (i.e, the case $\ell = L$) wipes out the upside-potential completely since the optimal strategy of the asset manager becomes "stop immediately" in this extreme case (i.e., $\ell = m = L$). Incidentally, a contract with a protection at some level is less valuable than an identical contract without a protection. The difference between these two values gives the fair price of the investors' protection. The investors must pay this difference to the asset manager in order to compensate for the asset manager's lost potential revenues due to "suboptimal" termination of the contract in the presence of the protection.

In other words, the asset manager will be willing to provide the protection only if the difference between the expected total revenues with and without the protection is cleared by the investors.

Our model also sheds some light on the *default timing problem* of a single firm. Note that the lower boundary a of the optimal continuation region in the first problem's solution may be interpreted as the "optimal default time" of a CDO. Instead of the value of a portfolio, if X represents the market value of a firm subject to unexpected "bad news" (downward jumps), then the asset manager's first problem and its solution translate into the default and sale timing problem of the firm and its solution. An action (default or sale) is optimal if the value X of the firm leaves the optimal continuation region (a, b). It is optimal to default if X reaches $(0, a]$, and optimal to sell the firm if X reaches $[b, \infty)$. Our solution extends the work of Duffie [15, Chapter 11] who calculates (based on the paper by Leland [22]) the optimal default time for a single firm whose asset value is modeled by a geometric Brownian motion.

Let us also mention that optimal stopping problems (especially, pricing American-type options) for Lévy processes have been extensively studied; see, for example, Chan [5], Pham [26], Mordecki [25, 24], Boyarchenko and Levendroskiǐ [3], Kou and Wang [21] and Asmussen et al. [1].

The problems are formulated in Section 2. The solutions of first and second problems are studied in Sections 3 and 4, respectively. The solutions methods of Problems 1 and 2 are illustrated on a numerical example in Section 5.

2. The problem description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a Brownian motion $B = \{B_t, t \geq 0\}$ and an independent Poisson process $N = \{N_t, t \geq 0\}$ with the constant arrival rate λ, both adapted to some filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

An asset manager borrows L dollars from some investors and invests in some risky asset $X = \{X_t, t \geq 0\}$. The process X has the dynamics

$$\frac{dX_t}{X_{t-}} = (\mu - \delta)\mathrm{d}t + \sigma \mathrm{d}B_t - y_0\,[\mathrm{d}N_t - \lambda \mathrm{d}t], \quad t \geq 0 \tag{2.1}$$

for some constants $\mu \in \mathbb{R}$, $\sigma > 0$, $\delta > 0$ and $y_0 \in (0, 1)$. We denote by δ the dividend rate or the management fee received by the asset manager. Note that the absolute value of relative jump sizes equals y_0, and the jumps are downwards. Therefore, the asset price

$$X_t = X_0 \exp\left\{\left(\mu - \delta - \frac{1}{2}\sigma^2 + \lambda y_0\right)t + \sigma B_t\right\}(1 - y_0)^{N_t}, \qquad t \geq 0$$

is a geometric Brownian motion subject to downward jumps with constant relative jump sizes.

An interesting example of our setting is a portfolio of defaultable bonds as in the *collateralized debt obligations*. Let X_t be the value of a portfolio of k defaultable bonds. After every default, the portfolio loses y_0 percent of its market value. The default times of each bond i constitutes a Poisson process with the intensity rate λ_i independent of others. Therefore, defaults occur at the rate $\lambda \triangleq \sum_{i=1}^k \lambda_i$ at the level of the portfolio. The loss ratio upon a default is the same constant y_0 across the bonds. The defaulted bond is immediately sold at the market, and a bond with a similar default rate is bought using the sales proceeds. Under this assumption, defaults occur at the fixed rate λ because the number of bonds in the portfolio is fixed at k. Egami and Esteghamat [18] showed that the dynamics in (2.1) are a good approximation of the dynamics of the aggregate value of individual defaultable bonds when priced in the "intensity-based" modeling framework (see, e.g., Duffie and Singleton [17]). The jump size y_0 on the portfolio level has to be calibrated.

Suppose that the asset manager pays the investors a coupon of c percent on the face value of the initial borrowing L on a continuously compounded basis. We assume $c < \delta$. The asset manager has the right to terminate the contract at any time $\tau \in \mathbb{R}_+$ and receive $(X_\tau - L)^+$. Dividend and coupon payments to the parties cease upon the termination of the contract. Let $0 < r < c$ be the risk-free interest rate, and $\mathcal{S}$ be the collection of all $\mathbb{F}$-stopping times. The *asset manager's first problem* is to calculate her maximum expected discounted total income

$$U(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}_x^\gamma \left[e^{-r\tau}(X_\tau - L)^+ + \int_0^\tau e^{-rt}(\delta X_t - cL)\mathrm{d}t \right], \quad x \in \mathbb{R}_+, \tag{2.2}$$

where $\mathbb{E}^\gamma$ is taken under the equivalent martingale measure $\mathbb{P}^\gamma$ for a specified market price γ of the jump risk, and to find some $\tau^* \in \mathcal{S}$ that attains the supremum (if it exists) under the condition

$$0 < r < c < \delta.$$

In the case of real CDOs, the dividend payment is often subordinated to the coupon payment. But since we allow the possibility that the asset manager's net running cash flow $\delta X_t - cL$ becomes negative, our formulation has more stringent requirement on the asset manager than a simple subordination.

In the *asset manager's second problem*, the investors' assets have limited protection. In the presence of the limited *protection at level* $\ell > 0$, the contract terminates at time $\widetilde{\tau}_{(\ell,\infty)} \triangleq \inf\{t \geq 0 : X_t \notin (\ell, \infty)\}$ automatically. The asset manager wants to maximize her expected total discounted earnings as in (2.2), but now the supremum has to be taken over all $\mathbb{F}$-adapted stopping times $\tau \in \mathcal{S}$ which are less than or equal to $\widetilde{\tau}_{(\ell,\infty)}$ almost surely.

3. The solution of the asset manager's first problem

In the no-arbitrage pricing framework, the value of a contract contingent on the asset X is the expectation of the total discounted payoff of the contract under some equivalent martingale measure. Since the dynamics of X in (2.1) contain jumps, there are more than one equivalent martingale measure; see, e.g. Schoutens [27] and Nunno [14]. The restriction to $\mathcal{F}_t$ of every equivalent martingale measure $\mathbb{P}^\gamma$ in a large class admits a Radon-Nikodym derivative in the form of

$$\left.\frac{d\mathbb{P}^\gamma}{d\mathbb{P}}\right|_{\mathcal{F}_t} \triangleq \eta_t \quad \text{and} \quad \frac{d\eta_t}{\eta_{t-}} = \beta \mathrm{d}B_t + (\gamma - 1)[\mathrm{d}N_t - \lambda \mathrm{d}t], \quad t \geq 0 \quad (\eta_0 = 1), \tag{3.1}$$

which has the solution $\eta_t = \exp\left\{\beta B_t - \frac{1}{2}\beta^2 t + N_t \log\gamma - (\gamma - 1)\lambda t\right\}$, $t \geq 0$ for some constants $\beta \in \mathbb{R}$ and $\gamma > 0$. The constants β and γ are known as the market price of the diffusion risk and the market price of the jump risk, respectively, and satisfy the drift condition

$$\gamma > 0 \quad \text{and} \quad \mu - r + \sigma\beta - \lambda y_0(\gamma - 1) = 0. \tag{3.2}$$

Then the discounted value process $\{e^{-(r-\delta)t}X_t : t \geq 0\}$ before the dividends are paid is a $(\mathbb{P}^\gamma, \mathbb{F})$-martingale; see, e.g., Pham [26], Colwell and Elliott [6], Cont and Tankov [7]. Girsanov theorem implies that $B_t^\gamma \triangleq B_t - \beta t$, $t \geq 0$ is a standard Brownian motion, and N_t, $t \geq 0$ is a homogeneous Poisson process with intensity $\lambda\gamma$ independent of B^γ under the new measure $\mathbb{P}^\gamma$. Then

$$\frac{dX_t}{X_{t-}} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}B_t^\gamma - y_0\,[\mathrm{d}N_t - \lambda\gamma\mathrm{d}t], \quad t \geq 0, \tag{3.3}$$

where $\mu - \delta + \beta\sigma - \lambda y_0(\gamma - 1) = r - \delta$ follows from the drift condition in (3.2). Itô's rule implies

$$X_t = X_0 \exp\left\{\left(r - \delta - \frac{1}{2}\sigma^2 + \lambda\gamma y_0\right)t + \sigma B_t^\gamma\right\}(1-y_0)^{N_t}, \qquad t \geq 0. \tag{3.4}$$

The infinitesimal generator of the process X under the probability measure $\mathbb{P}^\gamma$ coincides with the second order differential-difference operator

$$(\mathcal{A}^\gamma f)(x) \triangleq (r - \delta + \lambda\gamma y_0)\,x\,f'(x) + \frac{1}{2}\,\sigma^2\,x^2\,f''(x) + \lambda\gamma[f(x(1-y_0)) - f(x)] \tag{3.5}$$

on the collection of twice-continuously differentiable functions $f(\cdot)$.

Because $\{e^{-(r-\delta)t}X_t, t \geq 0\}$ is a martingale under $\mathbb{P}^\gamma$, we have $\mathbb{E}_x^\gamma[\int_0^\infty \delta X_t e^{-rt}\mathrm{d}t] = \int_0^\infty \delta x e^{-\delta t}\mathrm{d}t = x$, and for every stopping time $\tau \in \mathcal{S}$, the strong Markov property implies that $\mathbb{E}_x^\gamma[\int_0^\tau \delta X_t e^{-rt}\mathrm{d}t] = \mathbb{E}_x^\gamma[\int_0^\infty \delta X_t e^{-rt}\mathrm{d}t] - \mathbb{E}_x^\gamma[\int_\tau^\infty \delta X_t e^{-rt}\mathrm{d}t] = x - \mathbb{E}_x^\gamma[e^{-r\tau}\mathbb{E}_{X_\tau}^\gamma(\int_0^\infty \delta X_s e^{-rs}\mathrm{d}s)] = x - \mathbb{E}_x^\gamma[e^{-r\tau}X_\tau]$, $x \in \mathbb{R}_+$. Because $\mathbb{E}_x^\gamma[\int_0^\tau cLe^{-rt}\mathrm{d}t] = \frac{cL}{r} - \mathbb{E}_x^\gamma[\frac{cL}{r}\,e^{-r\tau}]$ for every $\tau \in \mathcal{S}$ and $x \in \mathbb{R}_+$, (2.2) can be rewritten as

$$U(x) = V(x) + x - \frac{cL}{r},\ x \in \mathbb{R}_+, \quad \text{where} \quad V(x) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}_x^\gamma\left[e^{-r\tau}\left((X_\tau - L)^+ - X_\tau + \frac{cL}{r}\right)\right] \tag{3.6}$$

is a discounted optimal stopping problem with the terminal reward function

$$h(x) \triangleq (x-L)^+ - x + \frac{cL}{r}, \qquad x \in \mathbb{R}_+. \tag{3.7}$$

We fix the market price γ of jump risk, and the market price β is determined by the drift condition in (3.2). In the remainder, we shall describe the solution of the optimal stopping problem (3.6).

Let $T_1, T_2, \ldots$ be the arrival times of process N. Observe that $X_{T_{n+1}} = (1-y_0)X_{T_{n+1}-}$ and

$$\frac{X_{T_n+t}}{X_{T_n}} = \exp\left\{\left(r - \delta + \lambda\gamma y_0 - \frac{\sigma^2}{2}\right)t + \sigma(B_{T_n+t}^\gamma - B_{T_n}^\gamma)\right\}, \quad 0 \leq t < T_{n+1} - T_n,\ n \geq 1.$$

Le us define for every $n \geq 0$ the standard Brownian motion $B_t^{\gamma,n} := B_{T_n+t}^\gamma - B_{T_n}^\gamma$, $t \geq 0$ and Poisson process $T_k^{(n)} := T_{n+k} - T_n$, $k \geq 0$, respectively, under $\mathbb{P}^\gamma$ and the one-dimensional diffusion process

$$Y_t^{y,n} \triangleq y \exp\left\{\left(r - \delta + \lambda\gamma y_0 - \frac{\sigma^2}{2}\right)t + \sigma B_t^{\gamma,n}\right\},\ t \geq 0, \tag{3.8}$$

which has dynamics

$$Y_0^{y,n} = y \quad \text{and} \quad \mathrm{d}Y_t^{y,n} = Y_t^{y,n}\left[(r - \delta + \lambda\gamma y_0)\mathrm{d}t + \sigma\mathrm{d}B_t^{\gamma,n}\right], \quad t \geq 0 \tag{3.9}$$

and infinitesimal generator (under $\mathbb{P}_x^\gamma$)

$$(\mathcal{A}_0^\gamma f)(y) = \frac{\sigma^2 y^2}{2} f''(y) + (r - \delta + \lambda\gamma y_0) y f'(y) \tag{3.10}$$

acting on twice-continuously differentiable functions $f : \mathbb{R}_+ \mapsto \mathbb{R}$. Then X coincides with $Y^{X_{T_n},n}$ on $[T_n, T_{n+1})$ and jumps to $(1-y_0)Y^{X_{T_n},n}_{T_{n+1}-T_n}$ at time T_{n+1} for every $n \geq 0$; namely,

$$X_{T_n+t} = \begin{cases} Y^{X_{T_n},n}_t, & 0 \leq t < T_{n+1} - T_n, \\ (1-y_0)Y^{X_{T_n},n}_{T_{n+1}-T_n}, & t = T_{n+1} - T_n. \end{cases}$$

For $n = 0$, we shall write $Y^{y,0} \equiv Y^y = y \exp\left\{(r - \delta - \lambda\gamma y_0 - \sigma^2/2)t + \sigma B^\gamma_t\right\}$ and $Y^{X_0,0} \equiv Y^{X_0}$.

3.1. **A dynamic programming operator.** Let $\mathcal{S}_B$ denote the collection of all stopping times of the diffusion process Y^{X_0}, or equivalently, Brownian motion B. Let us take any arbitrary but fixed stopping time $\tau \in \mathcal{S}_B$ and consider the following stopping strategy toward the solution of (3.6):

(i) on $\{\tau < T_1\}$ stop at time τ,

(ii) on $\{\tau \geq T_1\}$, update X at time T_1 to $X_{T_1} = (1-y_0)Y^{X_0}_{T_1}$ and continue optimally thereafter.

The value of this new strategy is $\mathbb{E}^\gamma_x[e^{-r\tau}h(X_\tau)1_{\{\tau<T_1\}} + e^{-rT_1}V(X_{T_1})1_{\{\tau\geq T_1\}}]$ and equals

$$\mathbb{E}^\gamma_x\left[e^{-r\tau}h(Y^{X_0}_\tau)1_{\{\tau<T_1\}} + e^{-rT_1}V((1-y_0)Y^{X_0}_{T_1})1_{\{\tau\geq T_1\}}\right] \\ = \mathbb{E}^\gamma_x\left[e^{-(r+\lambda\gamma)\tau}h(Y^{X_0}_\tau) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t}V((1-y_0)Y^{X_0}_t)\mathrm{d}t\right].$$

If for every bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we introduce the operator

$$(Jw)(x) \triangleq \sup_{\tau\in\mathcal{S}_B} \mathbb{E}^\gamma_x\left[e^{-(r+\lambda\gamma)\tau}h(Y^{X_0}_\tau) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t}w((1-y_0)Y^{X_0}_t)\mathrm{d}t\right], \quad x \geq 0, \tag{3.11}$$

then we expect the value function $V(\cdot)$ of (3.6) to be the unique fixed point of operator J; namely, $V(\cdot) = (JV)(\cdot)$, and that $V(\cdot)$ is the pointwise limit of the successive approximations

$$\begin{aligned} v_0(x) &\triangleq h(x) = (x-L)^+ - x + \frac{cL}{r}, && x \geq 0, \\ v_n(x) &\triangleq (Jv_{n-1})(x), && x \geq 0,\ n \geq 1. \end{aligned}$$

Lemma 1. *Let $w_1, w_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ be bounded. If $w_1(\cdot) \leq w_2(\cdot)$, then $(Jw_1)(\cdot) \leq (Jw_2)(\cdot)$. If $w(\cdot)$ is nonincreasing convex function such that $h(\cdot) \leq w(\cdot) \leq cL/r$, then $(Jw)(\cdot)$ has the same properties.*

The proof easily follows from the linearity of $y \mapsto Y^y_t$ for every fixed $t \geq 0$ and the definition of the operator J. The next proposition guarantees the existence of unique fixed point of J.

Proposition 2. *For every bounded $w_1, w_2 : \mathbb{R}_+ \mapsto \mathbb{R}$, we have $\|Jw_1 - Jw_2\| \leq \frac{\lambda\gamma}{r+\lambda\gamma}\|w_1 - w_2\|$, where $\|w\| = \sup_{x\in\mathbb{R}_+}|w(x)|$; namely, J acts as a contraction mapping on the bounded functions.*

Proof. Because $w_1(\cdot), w_2(\cdot)$ are bounded, $(Jw_1)(\cdot)$ and $(Jw_2)(\cdot)$ are finite, and for every ε and $x > 0$, there are ε-optimal stopping times $\tau_1(\varepsilon, x)$ and $\tau_2(\varepsilon, x)$, which may depend on ε and x, such that

$$(Jw_i)(x) - \varepsilon \leq \mathbb{E}^\gamma_x\Big[e^{-(r+\lambda\gamma)\tau_i(\varepsilon,x)}h(Y^{X_0}_{\tau_i(\varepsilon,x)}) + \int_0^{\tau_i(\varepsilon,x)} \lambda\gamma e^{-(r+\lambda\gamma)t}w_i((1-y_0)Y^{X_0}_t)\mathrm{d}t\Big], \quad i = 1, 2.$$

Therefore, $(Jw_1)(x) - (Jw_2)(x) \leq \varepsilon + \|w_1 - w_2\| \int_0^\infty \lambda\gamma e^{-(r+\lambda\gamma)t}\mathrm{d}t = \varepsilon + \|w_1 - w_2\|\frac{\lambda\gamma}{r+\lambda\gamma}$. Interchanging the roles of $w_1(\cdot)$ and $w_2(\cdot)$ gives $|(Jw_1)(x) - (Jw_2)(x)| \leq \varepsilon + \|w_1 - w_2\|\frac{\lambda\gamma}{r+\lambda\gamma}$ for every $x > 0$ and $\varepsilon > 0$. Taking the supremum of both sides over $x \geq 0$ completes the proof. $\square$

Lemma 3. *The sequence* $(v_n)_{n\geq 0}$ *of successive approximations is nondecreasing. Therefore, the pointwise limit* $v_\infty(x) \triangleq \lim_{n\to\infty} v_n(x)$, $x \geq 0$ *exists. Every* $v_n(\cdot)$, $n \geq 0$ *and* $v_\infty(\cdot)$ *are nonincreasing, convex, and bounded between* $h(\cdot)$ *and* cL/r.

Lemma 3 follows from repeated applications of Lemma 1. Proposition 4 below shows that the unique fixed point of J is the uniform limit of successive approximations.

Proposition 4. *The limit* $v_\infty(\cdot) = \lim_{n\to\infty} v_n(\cdot) = \sup_{n\geq 0} v_n(\cdot)$ *is the unique bounded fixed point of operator* J. *Moreover,* $0 \leq v_\infty(x) - v_n(x) \leq \frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^n$ *for every* $x \geq 0$.

Proof. Since $v_n(\cdot) \nearrow v_\infty(\cdot)$ as $n \to \infty$, and every $v_n(\cdot)$ is bounded from below by $\frac{c-r}{r}L$, and $\mathbb{E}^\gamma_\cdot\big[\int_0^\tau e^{-(r+\lambda\gamma)t}\frac{c-r}{r}L\mathrm{d}t\big] < \infty$ for every $\tau \in \mathcal{S}_B$, the monotone convergence theorem implies that

$$\begin{aligned} v_\infty(x) &= \sup_{n\geq 0} v_n(x) = \sup_{\tau\in\mathcal{S}_B}\lim_{n\to\infty}\mathbb{E}^\gamma_x\Big[e^{-(r+\lambda\gamma)\tau}h(Y^{X_0}_\tau) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t}v_n((1-y_0)Y^{X_0}_t)\mathrm{d}t\Big] \\ &= \sup_{\tau\in\mathcal{S}_B}\mathbb{E}^\gamma_x\Big[e^{-(r+\lambda\gamma)\tau}h(Y^{X_0}_\tau) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t}v_\infty((1-y_0)Y^{X_0}_t)\mathrm{d}t\Big] = (Jv_\infty)(x). \end{aligned}$$

Thus, $v_\infty(\cdot)$ is the bounded fixed point of contraction mapping J. Lemma 3 implies $0 \leq v_\infty(\cdot)-v_n(\cdot)$, and $\|v_\infty - v_n\| = \|Jv_\infty - Jv_{n-1}\| \leq \frac{\lambda\gamma}{r+\lambda\gamma}\|v_\infty - v_{n-1}\| \leq \ldots \leq (\frac{\lambda\gamma}{r+\lambda\gamma})^n\frac{cL}{r}$ for every $n \geq 1$. □

3.2. **The solution of the optimal stopping problem in (3.11).** We shall next solve the optimal stopping problem Jw in (3.11) for every fixed $w : \mathbb{R}_+ \mapsto \mathbb{R}$ which satisfies the following assumption:

Assumption 5. *Let* $w : \mathbb{R}_+ \mapsto \mathbb{R}$ *be nonincreasing, convex, bounded between* $h(\cdot)$ *and* cL/r, *and* $w(+\infty) = \frac{c-r}{r}L$ *and* $w(0+) = \frac{c}{r}L$.

We shall calculate the value function $(Jw)(\cdot)$ and explicitly identify an optimal stopping rule. Because $w(\cdot)$ is bounded, we have

$$\mathbb{E}^\gamma_x\left[\int_0^\infty e^{-(r+\lambda\gamma)t}|w((1-y_0)Y^{X_0}_t)|\mathrm{d}t\right] \leq \|w\|\int_0^\infty e^{-(r+\lambda\gamma)t}\mathrm{d}t = \frac{\|w\|}{r+\lambda\gamma} < \infty, \quad x \geq 0,$$

and for every stopping time $\tau \in \mathcal{S}_B$, the strong Markov property of Y^{X_0} at time τ implies that

$$\begin{aligned} (Hw)(x) &\triangleq \mathbb{E}^\gamma_x\left[\int_0^\infty e^{-(r+\lambda\gamma)t}w((1-y_0)Y^{X_0}_t)\mathrm{d}t\right] \\ &= \mathbb{E}^\gamma_x\left[\int_0^\tau e^{-(r+\lambda\gamma)t}w((1-y_0)Y^{X_0}_t)\mathrm{d}t\right] + \mathbb{E}^\gamma_x\left[e^{-(r+\lambda\gamma)\tau}(Hw)(Y^{X_0}_\tau)\right]. \end{aligned} \tag{3.12}$$

Therefore, $\mathbb{E}^\gamma_x[\int_0^\tau e^{-(r+\lambda\gamma)t}w((1-y_0)Y^{X_0}_t)\mathrm{d}t] = (Hw)(x) - \mathbb{E}^\gamma_x[e^{-(r+\lambda\gamma)\tau}(Hw)(Y^{X_0}_\tau)]$, and we can write the expected payoff $\mathbb{E}^\gamma_x[e^{-(r+\lambda\gamma)\tau}h(Y^{X_0}_\tau) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t}w((1-y_0)Y^{X_0}_t)\mathrm{d}t]$ in (3.11) as $\lambda\gamma(Hw)(x) + \mathbb{E}^\gamma_x\left[e^{-(r+\lambda\gamma)\tau}\{h - \lambda\gamma(Hw)\}(Y^{X_0}_\tau)\right]$ for every $\tau \in \mathcal{S}_B$ and $x > 0$. If we define

$$(Gw)(x) \triangleq \sup_{\tau\in\mathcal{S}_B}\mathbb{E}^\gamma_x\left[e^{-(r+\lambda\gamma)\tau}\{h - \lambda\gamma(Hw)\}(Y^{X_0}_\tau)\right], \quad x > 0, \tag{3.13}$$

then the value function in (3.11) can be calculated by

$$(Jw)(x) = \lambda\gamma(Hw)(x) + (Gw)(x), \quad x > 0. \tag{3.14}$$

We take $\mathbb{R}_+ = [0, \infty)$ everywhere. The state 0 is a natural boundary point for the geometric Brownian motion Y: it starts from 0, then it stays there forever and cannot get into the interior of the state space with probability one. If it starts in the interior of its state space (namely, $Y_0 \in (0, \infty)$), then it can never reach 0. For all practical purposes, we can neglect state 0 and the values at 0 of any functions related to Y. For completeness, we can for example define $G(0) = G(0+)$, $(Hw)(0) = (Hw)(0+)$, and $(Jw)(0) = (Jw)(0+)$.

Let us first calculate $(Hw)(\cdot)$. Let $\psi(\cdot)$ and $\varphi(\cdot)$ be, respectively, the increasing and decreasing solutions of the second order ordinary differential equation $(\mathcal{A}_0 f)(y) - (r + \lambda\gamma)f(y) = 0$, $y > 0$ with boundary conditions, respectively, $\psi(0+) = 0$ and $\varphi(+\infty) = 0$, where $\mathcal{A}_0$ is the infinitesimal generator in (3.10) of diffusion process $Y^{X_0} \equiv Y^{X_0,0}$. One can easily check that

$$\psi(y) = y^{\alpha_1} \quad \text{and} \quad \varphi(y) = y^{\alpha_0} \quad \text{for every } y > 0, \tag{3.15}$$

with the Wronskian

$$W(y) = \psi'(y)\varphi(y) - \psi(y)\varphi'(y) = (\alpha_0 + \alpha_1)y^{\alpha_0+\alpha_1-1}, \quad y > 0, \tag{3.16}$$

where $\alpha_0 < \alpha_1$ are the roots of the characteristic function $g(\alpha) = \frac{\sigma^2}{2}\alpha(\alpha - 1) + (r - \delta + \lambda\gamma y_0)\alpha - (r + \lambda\gamma)$ of the above ordinary differential equation. Because both $g(0) < 0$ and $g(1) < 0$, we have

$$\alpha_0 < 0 < 1 < \alpha_1.$$

Let us denote the hitting and exit times of diffusion process Y^{X_0}, respectively, by

$$\begin{aligned} \tau_a &\triangleq \inf\{t \geq 0;\ Y_t^{X_0} = a\}, && a > 0, \\ \tau_{ab} &\triangleq \inf\{t \geq 0;\ Y_t^{X_0} \notin (a, b)\}, && 0 < a < b < \infty, \end{aligned}$$

and define operator

$$(H_{ab}w)(x) \triangleq \mathbb{E}_x^\gamma \left[\int_0^{\tau_{ab}} e^{-(r+\lambda\gamma)t} w((1 - y_0)Y_t^{X_0}) \mathrm{d}t + 1_{\{\tau_{ab} < \infty\}} e^{-(r+\lambda\gamma)\tau_{ab}} h(Y_{\tau_{ab}}^{X_0}) \right] \quad \text{and}$$

$$\psi_a(y) \triangleq \psi(y) - \frac{\psi(a)}{\varphi(a)}\varphi(y) \quad \text{and} \quad \varphi_b(y) \triangleq \varphi(y) - \frac{\varphi(b)}{\psi(b)}\psi(y) \quad \text{for every } y > 0,$$

which are, respectively, the increasing and decreasing solutions of $(\mathcal{A}_0 f)(y) - (r + \lambda\gamma)f(y) = 0$, $a < y < b$ with boundary conditions, respectively, $f(a) = 0$ and $f(b) = 0$. In terms of $W(\cdot)$ in (3.16), the Wronskian of $\psi_a(\cdot)$ and $\varphi_a(\cdot)$ becomes

$$W_{ab}(y) = \psi_a'(y)\varphi_b(y) - \psi_a(y)\varphi_b'(y) = \left[1 - \frac{\psi(a)}{\varphi(a)}\frac{\varphi(b)}{\varphi(b)} \right] W(y), \quad y > 0. \tag{3.17}$$

Taylor and Karlin [20, Chapter 15], Borodin and Salminen [2] prove Lemma 6 below.

Lemma 6. *For every $x > 0$, we have*

$$\begin{aligned} (Hw)(x) &\triangleq \mathbb{E}_x^\gamma \left[\int_0^\infty e^{-(r+\lambda\gamma)t} w((1 - y_0)Y_t^{X_0}) \mathrm{d}t \right] = \lim_{a\downarrow 0, b\uparrow\infty} (H_{ab}w)(x) \\ &= \varphi(x) \int_0^x \frac{2\psi(\xi)w((1 - y_0)\xi)}{p^2(\xi)W(\xi)} \mathrm{d}\xi + \psi(x) \int_x^\infty \frac{2\varphi(\xi)w((1 - y_0)\xi)}{p^2(\xi)W(\xi)} \mathrm{d}\xi, \end{aligned}$$

which is twice-continuously differentiable on $\mathbb{R}_+$ *and satisfies the ordinary differential equation* $(\mathcal{A}_0 f)(x) - (r+\lambda\gamma)f(x) + w((1-y_0)x) = 0$.

Using the potential theoretic direct methods of Dayanik and Karatzas [11] and Dayanik [10], we shall now solve the optimal stopping problem $(Gw)(\cdot)$ (3.13) with payoff function $(h - \lambda\gamma(Hw))(x) =$

$$(x-L)^+ - x + \frac{cL}{r} - \lambda\gamma\Big[\varphi(x)\int_0^x \frac{2\psi(\xi)w((1-y_0)\xi)}{p^2(\xi)W(\xi)}\mathrm{d}\xi + \psi(x)\int_x^\infty \frac{2\varphi(\xi)w((1-y_0)\xi)}{p^2(\xi)W(\xi)}\mathrm{d}\xi\Big]$$
$$= (x-L)^+ - x + \frac{cL}{r} - \frac{2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)}\Big[x^{\alpha_0}\int_0^x \xi^{-1-\alpha_0}w((1-y_0)\xi)\mathrm{d}\xi + x^{\alpha_1}\int_x^\infty \xi^{-1-\alpha_1}w((1-y_0)\xi)\mathrm{d}\xi\Big],$$

where $\psi(x) = x^{\alpha_1}$, $\varphi(x) = x^{\alpha_0}$, $p^2(\xi) = \sigma^2\xi^2$, $W(\xi) = \psi'(\xi)\varphi(\xi) - \psi(\xi)\varphi'(\xi) = (\alpha_1-\alpha_0)\xi^{\alpha_0+\alpha_1-1}$.

We observe that $0 \leq (Hw)(x) = \mathbb{E}_x^\gamma[\int_0^\infty e^{-(r+\lambda\gamma)t}w((1-y_0)Y_t^{X_0})\mathrm{d}t] \leq \frac{cL}{r}\int_0^\infty e^{-(r+\lambda\gamma)t}\mathrm{d}t = \frac{cL}{r(r+\lambda\gamma)} < \infty$. Hence, $(h-\lambda\gamma(Hw))(\cdot)$ is bounded, and because $\psi(+\infty) = \varphi(0+) = +\infty$, we have

$$\limsup_{x\downarrow 0}\frac{(h-\lambda\gamma(Hw))^+(x)}{\varphi(x)} = 0 \qquad \text{and} \qquad \limsup_{x\uparrow\infty}\frac{(h-\lambda\gamma(Hw))^+(x)}{\psi(x)} = 0.$$

By Propositions 5.10 and 5.13 of Dayanik and Karatzas [11], value function $(Gw)(\cdot)$ is finite; the set

$$\Gamma[w] \triangleq \{x>0;\ (Gw)(x) = (h-\lambda\gamma(Hw))(x)\} = \{x>0;\ (Jw)(x) = h(x)\} \tag{3.18}$$

is the optimal stopping region, and

$$\tau[w] \triangleq \inf\{t \geq 0;\ Y_t^{X_0} \in \Gamma[w]\} \tag{3.19}$$

is an optimal stopping time for (3.13)—and for (3.11) because of (3.14). According to Proposition 5.12 of Dayanik and Karatzas [11], we have

$$(Gw)(x) = \varphi(x)(Mw)(F(x)),\ x\geq 0, \quad \text{and} \quad \Gamma[w] = F^{-1}(\{\zeta>0;\ (Mw)(\zeta) = (Lw)(\zeta)\}),$$

where $F(x) \triangleq \psi(x)/\varphi(x)$ and $(Mw)(\cdot)$ is the smallest nonnegative concave majorant on $\mathbb{R}_+$ of

$$(Lw)(\zeta) \triangleq \begin{cases} \dfrac{h-\lambda\gamma(Hw)}{\varphi}\circ F^{-1}(\zeta), & \zeta>0,\\ 0, & \zeta = 0.\end{cases} \tag{3.20}$$

To describe explicitly the form of the smallest nonnegative concave majorant $(Mw)(\cdot)$ of $(Lw)(\cdot)$, we shall firstly identify a few useful properties of function $(Lw)(\cdot)$. Because $Y^{X_0} \equiv X_0 Y^1$ by (3.8) and $w(\cdot)$ is bounded, the bounded convergence theorem implies that

$$\lim_{x\uparrow\infty}(Hw)(x) = \mathbb{E}_1^\gamma\left[\int_0^\infty e^{-(r+\lambda\gamma)t}\lim_{x\uparrow\infty}w((1-y_0)xY_t^1)\mathrm{d}t\right] = \frac{w(+\infty)}{r+\lambda\gamma} \leq \frac{cL}{r},$$

and $\lim_{x\uparrow\infty}(h-\lambda\gamma(Hw))(x) = \lim_{x\uparrow\infty}((x-L)^+ - x + \frac{cL}{r} - \lambda\gamma(Hw)(x)) \geq \frac{c-r}{r+\lambda\gamma}L > 0$. Therefore,

$$(Lw)(+\infty) = \lim_{x\uparrow\infty}\frac{(h-\lambda\gamma(Hw))(x)}{\varphi(x)} = +\infty. \tag{3.21}$$

Note also that

$$(Lw)'(\zeta) = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{h-\lambda\gamma(Hw)}{\varphi}\circ F^{-1}(\zeta)\right) = \left[\frac{1}{F'}\left(\frac{h-\lambda\gamma(Hw)}{\varphi}\right)'\right]\circ F^{-1}(\zeta).$$

Because $F(\cdot)$ is strictly increasing, we have $F' > 0$. Because $w(\cdot)$ is nonincreasing, the mapping $x \mapsto \mathbb{E}_x^\gamma[\int_0^\infty e^{-(r+\lambda\gamma)t} w((1-y_0)Y_t^{X_0})\mathrm{d}t] = \mathbb{E}_x^\gamma[\int_0^\infty e^{-(r+\lambda\gamma)t} w((1-y_0)X_0 Y_t^1)\mathrm{d}t]$ is decreasing. Then for $x > L$, because $h(\cdot) \equiv cL/r$ is constant, the mapping $x \mapsto (\frac{h-\lambda\gamma(Hw)}{\varphi})(x)$ is increasing.

For every $0 < x < L$, we can calculate explicitly that $[\frac{1}{F'}(\frac{h-\lambda\gamma(Hw)}{\varphi})'](x) =$

$$\frac{x^{-\alpha_1}}{\alpha_1 - \alpha_0}\left[(-\alpha_0)\frac{cL}{r} - (1-\alpha_0)x - \frac{\lambda\gamma(-\alpha_0)\alpha_1}{r+\lambda\gamma} x^{\alpha_1}\int_x^\infty \xi^{-1-\alpha_1} w((1-y_0)\xi)\mathrm{d}\xi\right],$$

and because $\lim_{x\downarrow 0} x^{\alpha_1}\int_x^\infty \xi^{-1-\alpha_1} w((1-y_0)\xi)\mathrm{d}\xi = \frac{w(0+)}{\alpha_1}$ and $\alpha_1 > 1$, we have

$$\lim_{x\downarrow 0}\left[\frac{1}{F'}\left(\frac{h-\lambda\gamma(Hw)}{\varphi}\right)'\right](x) = +\infty.$$

Let us also study the sign of the second derivative $(Lw)''(\cdot)$. For every $x \neq L$, Dayanik and Karatzas [11, page 192] show that

$$(Lw)''(F(x)) = \frac{2\varphi(x)}{p^2(x)W(x)F'(x)}(\mathcal{A}_0 - (r+\lambda\gamma))(h - \lambda\gamma(Hw))(x) \tag{3.22}$$

and $\varphi(\cdot), p^2(\cdot), W(\cdot), F'(\cdot)$ are positive. Therefore,

$$\mathrm{sgn}[(Lw)''(F(x))] = \mathrm{sgn}[(\mathcal{A}_0 - (r+\lambda\gamma))(h-\lambda\gamma(Hw))(x)].$$

Recall from Lemma 6 that $(\mathcal{A}_0 - (r+\lambda\gamma))(Hw)(x) = -w((1-y_0)x)$ and because $h(x) = (-x + \frac{cL}{r})1_{\{x\le L\}} + \frac{(c-r)L}{r}1_{\{x>L\}}$, we have $(\mathcal{A}_0 - (r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) =$

$$\left[\lambda\gamma(1-y_0)x - (r+\lambda\gamma)\frac{cL}{r} + \lambda\gamma w((1-y_0)x)\right]1_{\{x\le L\}} + \left[\lambda\gamma w((1-y_0)x) - (r+\lambda\gamma)\frac{(c-r)L}{r}\right]1_{\{x>L\}}.$$

Note that $\lim_{x\downarrow 0}(\mathcal{A}_0 - (r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) = -cL < 0$ and $\lim_{x\uparrow\infty}(\mathcal{A}_0 - (r+\lambda\gamma))(h - \lambda\gamma(Hw))(x) = -(c-r)L < 0$. Note also that $x \mapsto (\mathcal{A}_0 - (r+\lambda\gamma))(h-\lambda\gamma(Hw))(x)$ is convex and continuous on $x \in (0,L)$ and $x \in (L,\infty)$. Therefore, $(\mathcal{A}_0 - (r+\lambda\gamma))(h-\lambda\gamma(Hw))(x)$ is strictly negative in some open neighborhoods of 0 and $+\infty$, and in the complement of their unions, whose closure contains L if it is not empty, it is nonnegative. Therefore, (3.22) implies that $(Lw)(\zeta)$ is strictly concave in some neighborhood of $\zeta = 0$ and $\zeta = \infty$, and in the complement of their unions, whose closure contains $F(L)$ if it is not empty, this function is convex. Earlier we also showed that $\zeta \mapsto (Lw)(\zeta)$ is increasing at every $\zeta > F(L)$ and $(Lw)(+\infty) = (Lw)'(0+) = +\infty$. Moreover,

$$(Lw)'(F(L)-) - (Lw)'(F(L)+) = -\frac{L^{1-\alpha_1}}{\alpha_1 - \alpha_0} < 0;$$

namely, $(Lw)'(F(L)-) < (Lw)'(F(L)+)$. Two possible forms of $\zeta \mapsto (Lw)(\zeta)$ and their smallest nonnegative concave majorants $\zeta \mapsto (Mw)(\zeta)$ are depicted by two pictures of Figure 1.

The properties of the mapping $\zeta \mapsto (Lw)(\zeta)$ imply that there are unique numbers $0 < \zeta_1[w] < F(L) < \zeta_2[w] < \infty$ such that

$$(Lw)'(\zeta_1[w]) = \frac{(Lw)(\zeta_2[w]) - (Lw)(\zeta_1[w])}{\zeta_2[w] - \zeta_1[w]} = (Lw)'(\zeta_2[w]),$$

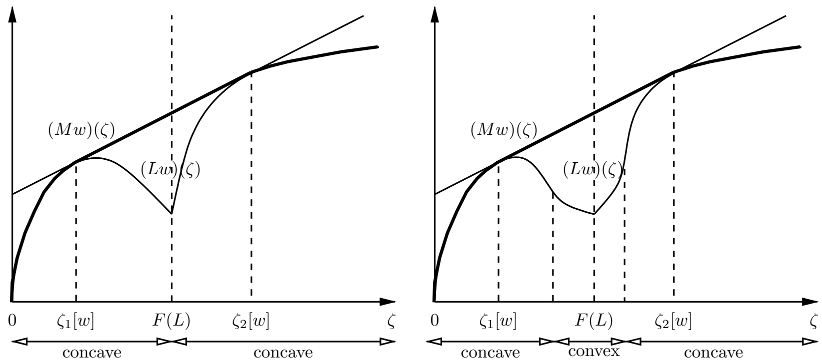

FIGURE 1. Two possible forms of $(Lw)(\cdot)$ and their smallest nonnegative concave majorants $(Mw)(\cdot)$.

and the smallest nonnegative concave majorant $(Mw)(\cdot)$ of $(Lw)(\cdot)$ on $(0, \zeta_1[w]] \cup [\zeta_2[w], \infty)$ coincides with $(Lw)(\cdot)$, and on $(\zeta_1[w], \zeta_2[w])$ with the straight-line that majorizes $(Lw)(\cdot)$ everywhere on $\mathbb{R}_+$ and is tangent to $(Lw)(\cdot)$ exactly at $\zeta = \zeta_1[w]$ and $\zeta_2[w]$; see Figure 1. More precisely,

$$(Mw)(\zeta) = \begin{cases} (Lw)(\zeta), & \zeta \in (0, \zeta_1[w]] \cup [\zeta_2[w], \infty), \\ \dfrac{\zeta_2[w] - \zeta}{\zeta_2[w] - \zeta_1[w]}(Lw)(\zeta_1[w]) + \dfrac{\zeta - \zeta_1[w]}{\zeta_2[w] - \zeta_1[w]}(Lw)(\zeta_2[w]), & \zeta \in (\zeta_1[w], \zeta_2[w]). \end{cases}$$

Let us define $x_1[w] \triangleq F^{-1}(\zeta_1[w])$ and $x_2[w] \triangleq F^{-1}(\zeta_2[w])$. Then by Proposition 5.12 of Dayanik and Karatzas [11], the value function of the optimal stopping problem in (3.13) equals

$$(3.23)\quad (Gw)(x) = \varphi(x)(Mw)(F(x))$$
$$= \begin{cases} (h - \lambda\gamma(Hw))(x), & x \in (0, x_1[w]] \cup [x_2[w], \infty), \\ \quad \dfrac{(x_2[w])^{\alpha_1-\alpha_0} - x^{\alpha_1-\alpha_0}}{(x_2[w])^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hw))(x_1[w]) & \\ +\dfrac{x^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}}{(x_2[w])^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hw))(x_2[w]), & x \in (x_1[w], x_2[w]). \end{cases}$$

The optimal stopping region in (3.18) becomes $\Gamma[w] = \{x > 0;\ (Gw)(x) = (h - \lambda\gamma)(Hw)(x)\} = (0, x_1[w]] \cup [x_2[w], \infty)$, and the optimal stopping time in (3.19) becomes

$$\tau[w] = \inf\{t \geq 0;\ Y_t^{X_0} \in (0, x_1[w]] \cup [x_2[w], \infty)\}.$$

Proposition 7. *The value function $x \mapsto (Gw)(\cdot)$ of (3.13) is continuously differentiable on $\mathbb{R}_+$ and twice-continuously differentiable on $\mathbb{R}_+ \setminus \{x_1[w], x_2[w]\}$. Moreover, $(Gw)(\cdot)$ satisfies*

(i) $(\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) = 0,\quad x \in (x_1[w], x_2[w]),$

(ii) $(Gw)(x) > h(x) - \lambda\gamma(Hw)(x),\quad x \in (x_1[w], x_2[w]),$

(iii) $(\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) < 0,\quad x \in (0, x_1[w]) \cup (x_2[w], \infty),$

(iv) $(Gw)(x) = h(x) - \lambda\gamma(Hw)(x),\quad x \in (0, x_1[w]] \cup [x_2[w], \infty).$

The differentiability of $(Gw)(\cdot)$ is clear from (3.23). The variational inequalities can be verified directly. For (iii) note that, if $x \in (0, x_1[w]) \cup (x_2[w], \infty)$, then $\mathrm{sgn}\{(\mathcal{A}_0 - (r+\lambda\gamma))(Gw)(x)\} = \mathrm{sgn}\{(\mathcal{A}_0 - (r+\lambda\gamma))(h - \lambda\gamma(Hw))(x)\} = \mathrm{sgn}\{(Lw)''(F(x))\} < 0$.

Because $(Hw)(\cdot)$ is twice-continuously differentiable and $(\mathcal{A}_0 - (r+\lambda\gamma)(Hw))(x) = -w((1-y_0)x)$ for every $x > 0$ by Proposition 6, Proposition 7 and (3.14) lead directly to the next proposition.

Proposition 8. *The value function $x \mapsto (Jw)(\cdot)$ of (3.11) is continuously differentiable on $\mathbb{R}_+$ and twice-continuously differentiable on $\mathbb{R}_+ \setminus \{x_1[w], x_2[w]\}$. Moreover, $(Jw)(\cdot)$ satisfies*

$$
\begin{aligned}
&(i) && (\mathcal{A}_0 - (r+\lambda\gamma))(Jw)(x) + \lambda\gamma w((1-y_0)x) = 0, && x \in (x_1[w], x_2[w]),\\
&(ii) && (Jw)(x) > h(x), && x \in (x_1[w], x_2[w]),\\
&(iii) && (\mathcal{A}_0 - (r+\lambda\gamma))(Jw)(x) + \lambda\gamma w((1-y_0)x) < 0, && x \in (0, x_1[w]) \cup (x_2[w], \infty),\\
&(iv) && (Jw)(x) = h(x), && x \in (0, x_1[w]] \cup [x_2[w], \infty).
\end{aligned}
$$

By Lemma 3, every $v_n(\cdot)$, $n \ge 0$ and $v_\infty(\cdot)$ are nonincreasing, convex, and bounded between $h(\cdot)$ and cL/r. Moreover, by using induction on n, we can easily show that $v_n(0+) = cL/r$ and $v_n(+\infty) = (c-r)L/r$ for every $n \in \{0, 1, \ldots, \infty\}$. Therefore, Proposition 8, applied to $w = v_\infty$, and Proposition 4 directly lead to the next theorem.

Theorem 9. *The function $x \mapsto v_\infty(x) = (Jv_\infty)(x)$ is continuously differentiable on $\mathbb{R}_+$ and twice-continuously differentiable on $\mathbb{R}_+ \setminus \{x_1[v_\infty], x_2[v_\infty]\}$ and satisfies the variational inequalities*

$$
\begin{aligned}
&(i) && (\mathcal{A}_0 - (r+\lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1-y_0)x) = 0, && x \in (x_1[v_\infty], x_2[v_\infty]),\\
&(ii) && v_\infty(x) > h(x), && x \in (x_1[v_\infty], x_2[v_\infty]),\\
&(iii) && (\mathcal{A}_0 - (r+\lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1-y_0)x) < 0, && x \in (0, x_1[v_\infty]) \cup (x_2[v_\infty], \infty),\\
&(iv) && v_\infty(x) = h(x), && x \in (0, x_1[v_\infty]] \cup [x_2[v_\infty], \infty),
\end{aligned}
$$

which can be expressed in terms of the generator $\mathcal{A}^\gamma$ in (3.5) of the jump-diffusion process X as

$$
\begin{aligned}
&(i)' && (\mathcal{A}^\gamma - r)v_\infty(x) = 0, && x \in (x_1[v_\infty], x_2[v_\infty]),\\
&(ii)' && v_\infty(x) > h(x), && x \in (x_1[v_\infty], x_2[v_\infty]),\\
&(iii)' && (\mathcal{A}^\gamma - r)v_\infty(x) < 0, && x \in (0, x_1[v_\infty]) \cup (x_2[v_\infty], \infty),\\
&(iv)' && v_\infty(x) = h(x), && x \in (0, x_1[v_\infty]] \cup [x_2[v_\infty], \infty).
\end{aligned}
$$

The next theorem identifies the value function and an optimal stopping time for the optimal stopping problem in (3.6). For every $w : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying Assumption 5 let us denote by $\widetilde{\tau}[w]$ the stopping time of jump-diffusion process X defined by

$$\widetilde{\tau}[w] \triangleq \inf\{t \ge 0;\ X_t \in (0, x_1[w]] \cup [x_2[w], \infty)\}.$$

Theorem 10. *For every $x \in \mathbb{R}_+$, we have $V(x) = v_\infty(x) = \mathbb{E}_x^\gamma\left[e^{-r\widetilde{\tau}[v_\infty]} h(X_{\widetilde{\tau}[v_\infty]})\right]$, and $\widetilde{\tau}[v_\infty]$ is an optimal stopping time for (3.6).*

Proof. Let $\widetilde{\tau}_{ab} = \inf\{t \geq 0;\ X_t \in (0,a] \cup [b,\infty)\}$ for every $0 < a < b < \infty$. By Itô's rule, we have

$$e^{-r(t\wedge\tau\wedge\widetilde{\tau}_{ab})} v_\infty(X_{t\wedge\tau\wedge\widetilde{\tau}_{ab}}) = v_\infty(X_0) + \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s$$
$$+ \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs} v_\infty(X_s)\sigma X_s \mathrm{d}B_s^\gamma + \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs}[v_\infty((1-y_\infty)X_{s-}) - v_\infty(X_{s-})](\mathrm{d}N_s - \lambda\gamma \mathrm{d}s)$$

for every $t \geq 0$, $\tau \in \mathcal{S}$, and $0 < a < b < \infty$. Because $v_\infty(\cdot)$ and $v'_\infty(\cdot)$ are continuous and bounded on every compact subinterval of $(0,\infty)$, both stochastic integrals are square-integrable martingales, and taking expectations of both sides gives

$$\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau\wedge\widetilde{\tau}_{ab})} v_\infty(X_{t\wedge\tau\wedge\widetilde{\tau}_{ab}})] = v_\infty(x) + \mathbb{E}_x^\gamma\Big[\int_0^{t\wedge\tau\wedge\widetilde{\tau}_{ab}} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s\Big]. \tag{3.24}$$

Because $(\mathcal{A}^\gamma - r)v_\infty(\cdot) \leq 0$ and $v_\infty(\cdot) \geq h(\cdot)$ by the variational inequalities of Theorem 9, we have $\mathbb{E}^\gamma[e^{-r(t\wedge\tau\wedge\widetilde{\tau}_{ab})} v_\infty(X_{t\wedge\tau\wedge\widetilde{\tau}_{ab}})] \leq v_\infty(x)$ for every $t \geq 0$, $\tau \in \mathcal{S}$, and $0 < a < b < \infty$. Because $\lim_{a\downarrow 0, b\uparrow\infty} \widetilde{\tau}_{ab} = \infty$ a.s. and $h(\cdot)$ is continuous and bounded, we can take limits of both sides as $t \uparrow \infty$, $a \downarrow 0$, $b \uparrow \infty$ and use the bounded convergence theorem to get $\mathbb{E}^\gamma[e^{-r\tau} v_\infty(X_\tau)] \leq v_\infty(x)$ for every $\tau \in \mathcal{S}$. Taking supremum over all $\tau \in \mathcal{S}$ gives $V(x) = \sup_{\tau\in\mathcal{S}} \mathbb{E}^\gamma[e^{-r\tau} v_\infty(X_\tau)] \leq v_\infty(x)$.

In order to show the reverse inequality, we replace in (3.24) τ and $\widetilde{\tau}_{ab}$ with $\widetilde{\tau}[v_\infty]$. Because $(\mathcal{A}^\gamma - r)v_\infty(x) = 0$ for every $x \in (x_1[v_\infty], x_2[v_\infty])$ by Theorem 9 $(i)'$, $\mathbb{E}_x^\gamma[e^{-r(t\wedge\widetilde{\tau}[v_\infty])} v_\infty(X_{t\wedge\widetilde{\tau}[v_\infty]})] = v_\infty(x) + \mathbb{E}_x^\gamma[\int_0^{t\wedge\widetilde{\tau}[v_\infty]} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s] = v_\infty(x)$ for every $t \geq 0$. Since v_∞ is bounded and continuous, taking limits as $t \uparrow \infty$ and the bounded convergence give $v_\infty(x) = \mathbb{E}_x^\gamma[e^{-r\widetilde{\tau}[v_\infty]} v_\infty(X_{\widetilde{\tau}[v_\infty]})] = \mathbb{E}_x^\gamma[e^{-r\widetilde{\tau}[v_\infty]} h(X_{\widetilde{\tau}[v_\infty]})] \leq V(x)$ by Theorem 9 $(iv)'$, which completes the proof. □

Proposition 11. *The optimal stopping regions* $\Gamma[v_n] = \{x > 0;\ (Jv_n)(x) \leq h(x)\} = (0, x_1[v_n]] \cup [x_2[v_n], \infty)$, $n \in \{0,1,\ldots,\infty\}$ *are decreasing, and* $0 < x_1[v_\infty] \leq \ldots \leq x_1[v_1] \leq x_1[v_0] \leq L \leq x_2[v_0] \leq x_2[v_1] \leq \ldots \leq x_2[v_\infty] < \infty$. *Moreover,* $x_1[v_\infty] = \lim_{n\to\infty} x_1[v_n]$ *and* $x_2[v_\infty] = \lim_{n\to\infty} x_2[v_n]$.

The proof follows from the monotonicity of operator J and that $v_n(x) \uparrow v_\infty(x)$ as $n \to \infty$ uniformly in $x > 0$. The next proposition and its corollary identify the optimal expected reward and nearly optimal stopping strategies for the asset manager in the first problem.

Proposition 12. *For all* $n \geq 0$, *we have* $v_\infty(x) \leq \mathbb{E}_x^\gamma[e^{-r\widetilde{\tau}[v_n]} h(X_{\widetilde{\tau}[v_n]})] + \frac{cL}{r}(\frac{\lambda}{r+\lambda\gamma})^{n+1}$. *Hence, for every* $\varepsilon > 0$ *and* $n \geq 0$ *such that* $\frac{cL}{r}(\frac{\lambda}{r+\lambda\gamma})^{n+1} \leq \varepsilon$, *the stopping time* $\widetilde{\tau}[v_n]$ *is* ε*-optimal for (3.6).*

Proof. Recall that $\tilde{\tau}[v_n] = \inf\{t \geq 0;\ X_t \in \Gamma[v_n]\} = \inf\{t \geq 0;\ X_t \in (0, x_1[v_n]] \cup [x_2[v_n], \infty)$. If we replace τ and $\widetilde{\tau}_{ab}$ in (3.24) with $\widetilde{\tau}[v_n]$, then for every $t \geq 0$ we obtain $\mathbb{E}_x^\gamma[e^{-r(t\wedge\widetilde{\tau}[v_n])} v_\infty(X_{t\wedge\widetilde{\tau}[v_n]})] = v_\infty(x) + \mathbb{E}_x^\gamma[\int_0^{t\wedge\widetilde{\tau}[v_n]} e^{-rs}(\mathcal{A}^\gamma - r)v_\infty(X_s)\mathrm{d}s] = v_\infty(x)$, because, for every $0 < t < \widetilde{\tau}[v_n]$ we have $X_t \in (x_1[v_n], x_2[v_n]) \subseteq (x_1[v_\infty], x_2[v_\infty])$, at every element x of which $(\mathcal{A}^\gamma - r)v_\infty(x)$ equals 0 according to 9 $(i)'$. Because $v_\infty(\cdot)$ is continuous and bounded, taking limits as $t \uparrow \infty$ and the bounded convergence theorem give $v_\infty(x) = \mathbb{E}_x^\gamma[e^{-r\widetilde{\tau}[v_n]} v_\infty(X_{\widetilde{\tau}[v_n]})]$. Because $(Jv_n)(\cdot) = h(\cdot)$ on $\Gamma[v_n] \ni X_{\widetilde{\tau}[v_n]}$ on $\{\widetilde{\tau}[v_n] < \infty\}$, Proposition 4 implies

$$v_\infty(x) \leq \mathbb{E}_x^\gamma\Big[e^{-r\widetilde{\tau}[v_n]}\Big(v_{n+1}(X_{\widetilde{\tau}[v_n]}) + \frac{cL}{r}\Big(\frac{\lambda\gamma}{r+\lambda\gamma}\Big)^{n+1}\Big)\Big] \leq \mathbb{E}_x^\gamma\Big[e^{-r\widetilde{\tau}[v_n]}\Big((Jv_n)(X_{\widetilde{\tau}[v_n]})\Big)\Big]$$

$$+\frac{cL}{r}\Big(\frac{\lambda\gamma}{r+\lambda\gamma}\Big)^{n+1}=\mathbb{E}^\gamma_x\Big[e^{-r\widetilde{\tau}[v_n]}\Big(h(X_{\widetilde{\tau}[v_n]})\Big)\Big]+\frac{cL}{r}\Big(\frac{\lambda\gamma}{r+\lambda\gamma}\Big)^{n+1}. \quad \square$$

Corollary 13. *The maximum expected reward of the asset manager is given by* $U(x)=x-\frac{cL}{r}+V(x)=x-\frac{cL}{r}+v_\infty(x)$ *for every* $x\geq 0$. *The stopping rule* $\widetilde{\tau}[v_\infty]$ *is optimal, and* $\widetilde{\tau}[v_n]$ *is* ε*-optimal for every* $\varepsilon>0$ *and* $n\geq 0$ *such that* $\frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^{n+1}<\varepsilon$: $U(x)=\mathbb{E}^\gamma_x[e^{-r\widetilde{\tau}[v_\infty]}(X_{\widetilde{\tau}[v_\infty]}-L)^+ + \int_0^{\widetilde{\tau}[v_\infty]}e^{-rt}(\delta X_t-cL)\mathrm{d}t]$ *and* $U(x)-\varepsilon\leq\mathbb{E}^\gamma_x[e^{-r\widetilde{\tau}[v_n]}(X_{\widetilde{\tau}[v_n]}-L)^+ + \int_0^{\widetilde{\tau}[v_n]}e^{-rt}(\delta X_t-cL)\mathrm{d}t]$, $x>0$.

4. The solution of the asset manager's second problem

In the *asset manager's second problem*, the investors' assets have limited protection. In the presence of the limited *protection at level* $\ell>0$, the contract terminates at time $\widetilde{\tau}_{\ell,\infty}\triangleq\inf\{t\geq 0: X_t\notin(\ell,\infty)\}$ automatically. The asset manager wants to maximize her expected total discounted earnings as in (2.2), but now the supremum has to be taken over all stopping times $\tau\in\mathcal{S}$ which are less than or equal to $\widetilde{\tau}_{\ell,\infty}$ almost surely. Namely, we would like to solve the problem

$$U_\ell(x)\triangleq\sup_{\tau\in\mathcal{S}}\mathbb{E}^\gamma_x\Big[e^{-r(\widetilde{\tau}_{\ell,\infty}\wedge\tau)}(X_{\widetilde{\tau}_{\ell,\infty}\wedge\tau}-L)^+ + \int_0^{\widetilde{\tau}_{\ell,\infty}\wedge\tau}e^{-rt}(\delta X_t-cL)\mathrm{d}t\Big],\quad x\in\mathbb{R}_+. \tag{4.1}$$

If $\ell<x_1[v_\infty]$, then $U_\ell(x)=U(x)=\mathbb{E}^\gamma_x[e^{-r(\widetilde{\tau}[v_\infty])}(X_{\widetilde{\tau}_{\ell,\infty}[v_\infty]}-L)^+ + \int_0^{\widetilde{\tau}[v_\infty]}e^{-rt}(\delta X_t-cL)\mathrm{d}t]$ for every $x>0$. On the one hand, because for every $\tau\in\mathcal{S}$, $\widetilde{\tau}[v_\infty]\wedge\tau$ also belongs to $\mathcal{S}$, we have $U_\ell(x)\leq U(x)$. On the other hand, because $\ell\leq x_1[v_\infty]$, we have a.s. $\widetilde{\tau}[v_\infty]=\widetilde{\tau}_{\ell,\infty}\wedge\widetilde{\tau}[v_\infty]\in\mathcal{S}$ and $U_\ell(x)\geq\mathbb{E}^\gamma_x[e^{-r\widetilde{\tau}[v_\infty]}(X_{\widetilde{\tau}[v_\infty]}-L)^+ + \int_0^{\widetilde{\tau}[v_\infty]}e^{-rt}(\delta X_t-cL)\mathrm{d}t]=U(x)$ for every x. Therefore, $U_\ell(x)=U(x)$ for every $x>0$ if $\ell\leq x_1[v_\infty]$.

Assumption 14. *In the remainder, suppose that the protection level* ℓ *is such that* $x_1[v_\infty]<\ell\leq L$.

The strong Markov property of X can be used to similarly show that

$$U_\ell(x)=x-\frac{cL}{r}+V_\ell(x),\ x\geq 0,\quad\text{where}\quad V_\ell(x)\triangleq\sup_{\tau\in\mathcal{S}}\mathbb{E}^\gamma_x\Big[e^{-r(\widetilde{\tau}_{\ell,\infty}\wedge\tau)}h(X_{\widetilde{\tau}_{\ell,\infty}\wedge\tau})\Big],\ x>0 \tag{4.2}$$

is the discounted optimal stopping problem for the stopped jump-diffusion process $X_{\widetilde{\tau}_{\ell,\infty}\wedge t}$, $t\geq 0$ with the same terminal payoff function $h(\cdot)$ as in (3.7). Let us define the stopping time $\tau_{\ell,\infty}\triangleq\inf\{t\geq 0;\ Y^{X_0}_t\notin(\ell,\infty)\}$ of diffusion process Y^{X_0} and the operator

$$\begin{aligned}(J_\ell w)(x)&\triangleq\sup_{\tau\in\mathcal{S}_B}\mathbb{E}^\gamma_x\Big[e^{-r\tau}h(X_{\tau_{\ell,\infty}\wedge\tau})1_{\{\tau_{\ell,\infty}\wedge\tau<T_1\}}+e^{-rT_1}w(X_{T_1})1_{\{\tau_{\ell,\infty}\wedge\tau\geq T_1\}}\Big]\\&=\sup_{\tau\in\mathcal{S}_B}\mathbb{E}^\gamma_x\Big[e^{-(r+\lambda\gamma)(\tau_{\ell,\infty}\wedge\tau)}h(Y^{X_0}_{\tau_{\ell,\infty}\wedge\tau})+\int_0^{\tau_{\ell,\infty}\wedge\tau}\lambda\gamma e^{-(r+\lambda\gamma)t}w((1-y_0)Y^{X_0}_t)\mathrm{d}t\Big],\quad x\geq 0.\end{aligned} \tag{4.3}$$

We expect that $V_\ell(\cdot)=(JV_\ell)(\cdot)$; namely, that $V_\ell(\cdot)$ is one of the fixed points of operator J_ℓ. We can find one of the fixed points of J_ℓ by taking limit of successive approximations defined by

$$v_{\ell,0}(x)\triangleq h(x)\quad\text{and}\quad v_{\ell,n}(x)\triangleq(J_\ell v_{\ell,n-1})(x),\ n\geq 1,\quad x>0.$$

Lemmas 1 and 3 and Propositions 2 and 4 hold with obvious changes. Let $w:\mathbb{R}_+\mapsto\mathbb{R}$ be a function as in Assumption 5. Then

$$(J_\ell w)(x)=\lambda\gamma(Hw)(x)+(G_\ell w)(x),\quad x>0,\quad\text{where} \tag{4.4}$$

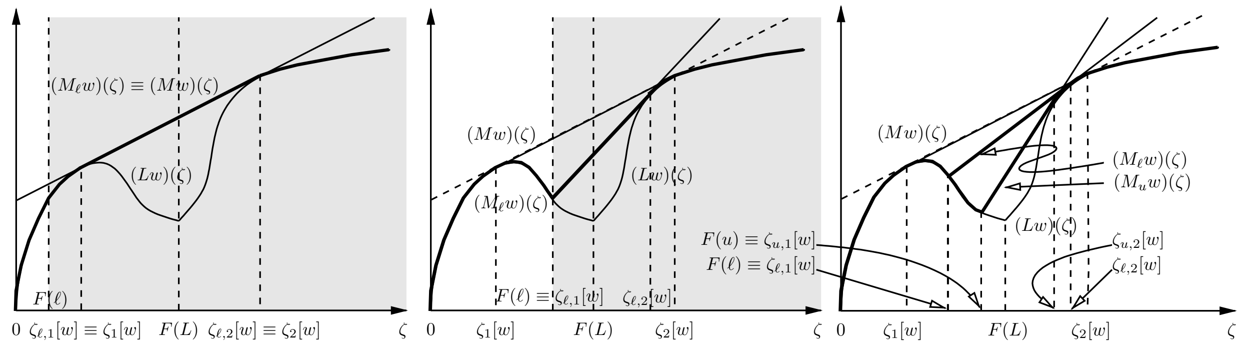

FIGURE 2. Sketches of $(Lw)(\cdot)$ and $(M_\ell w)(\cdot)$. On the left: $F(\ell) \le \zeta_1[w]$. In the middle: $\zeta_1[w] < F(\ell) \le F(L)$. On the right: the comparison of $(M_\ell w)(\cdot)$ and $(M_\ell w)(\cdot)$ for $0 < \ell < u < L$.

$$(4.5) \qquad (G_\ell w)(x) \triangleq \sup_{\tau \in \mathcal{S}_B} \mathbb{E}^\gamma_x \left[e^{-(r+\lambda\gamma)\tau_{\ell,\infty}\wedge\tau} \left\{ h - \lambda\gamma(Hw) \right\} (Y^{X_0}_{\tau_{\ell,\infty}\wedge\tau}) \right], \quad x > 0.$$

We obviously have $(G_\ell w)(x) = h(x) - \lambda\gamma(Hw)(x)$ for every $x \in (0, \ell]$. If the initial state X_0 of $Y^{X_0}_{\tau_{\ell,\infty}\wedge t}$, $t \ge 0$ is in (ℓ, ∞), then ℓ becomes an absorbing left-boundary for the stopped process $Y^{X_0}_{\tau_{\ell,\infty}\wedge t}$, $t \ge 0$.

Let $(M_\ell w)(\cdot)$ be the smallest concave majorant on $[F(\ell), \infty)$ of $(Lw)(\cdot)$ defined by (3.20) and equal on $(0, F(\ell))$ identically to $(Lw)(\cdot)$. Then by Proposition 5.5 of Dayanik and Karatzas [11] $(G_\ell w)(x) = \varphi(x)(M_\ell w)(F(x))$, $x > 0$ and $\Gamma_\ell[w] = F^{-1}(\{\zeta > 0;\ (M_\ell w)(\zeta) = (Lw)(\zeta)\})$ are value function and optimal stopping region for (4.5). The analysis of the shape of $(Lw)(\cdot)$ prior to Figure 1 implies that there are unique numbers $0 < \zeta_{\ell,1}[w] < F(L) < \zeta_{\ell,2}[w] < \infty$ such that

$$\begin{cases} \left\{ \begin{array}{c} (Lw)'(\zeta_{\ell,1}[w]) = \dfrac{(Lw)(\zeta_{\ell,2}[w]) - (Lw)(\zeta_{\ell,1}[w])}{\zeta_{\ell,2}[w] - \zeta_{\ell,1}[w]} = (Lw)'(\zeta_{\ell,2}[w]) \\ \text{namely,} \quad \zeta_{\ell,1}[w] \equiv \zeta_1[w] \quad \text{and} \quad \zeta_{\ell,2}[w] \equiv \zeta_2[w] \end{array} \right\} & \text{if } F(\ell) \le \zeta_1[w], \\ \zeta_{\ell,1}[w] = \ell \quad \text{and} \quad \dfrac{(Lw)(\zeta_{\ell,2}[w]) - (Lw)(\zeta_{\ell,1}[w])}{\zeta_{\ell,2}[w] - \zeta_{\ell,1}[w]} = (Lw)'(\zeta_{\ell,2}[w]) & \text{if } F(\ell) > \zeta_1[w], \end{cases}$$

and

$$(M_\ell w)(\zeta) = \begin{cases} (Lw)(\zeta), & \zeta \in (0, \zeta_{\ell,1}[w]] \cup [\zeta_{\ell,2}[w], \infty), \\ \dfrac{\zeta_{\ell,2}[w] - \zeta}{\zeta_{\ell,2}[w] - \zeta_{\ell,1}[w]}(Lw)(\zeta_{\ell,1}[w]) + \dfrac{\zeta - \zeta_{\ell,1}[w]}{\zeta_{\ell,2}[w] - \zeta_{\ell,1}[w]}(Lw)(\zeta_{\ell,2}[w]), & \zeta \in (\zeta_{\ell,1}[w], \zeta_{\ell,2}[w]). \end{cases}$$

Let us define $x_{\ell,1}[w] = F^{-1}(\zeta_{\ell,1}[w])$ and $x_{\ell,2}[w] = F^{-1}(\zeta_{\ell,2}[w])$. Then the value function equals

$$(4.6) \quad (G_\ell w)(x) = \varphi(x)(M_\ell w)(F(x))$$

$$= \begin{cases} (h - \lambda\gamma(Hw))(x), & x \in (0, x_{\ell,1}[w]] \cup [x_{\ell,2}[w], \infty), \\ \dfrac{(x_{\ell,2}[w])^{\alpha_1-\alpha_0} - x^{\alpha_1-\alpha_0}}{(x_{\ell,2}[w])^{\alpha_1-\alpha_0} - (x_{\ell,1}[w])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hw))(x_{\ell,1}[w]) & \\ \quad + \dfrac{x^{\alpha_1-\alpha_0} - (x_{\ell,1}[w])^{\alpha_1-\alpha_0}}{(x_{\ell,2}[w])^{\alpha_1-\alpha_0} - (x_{\ell,1}[w])^{\alpha_1-\alpha_0}}(h - \lambda\gamma(Hw))(x_{\ell,2}[w]), & x \in (x_{\ell,1}[w], x_{\ell,2}[w]) \end{cases}$$

and the optimal stopping region and an optimal stopping time are given by

$$\Gamma_\ell[w] = \{x > 0;\ (G_\ell w)(x) = (h - \lambda\gamma(Hw))(x)\} = (0, x_{\ell,1}[w]] \cup [x_{\ell,2}[w], \infty), \tag{4.7}$$

$$\tau_\ell[w] \triangleq \inf\{x > 0;\ Y_t^{X_0} \in \Gamma_\ell[w]\} = \inf\{x > 0;\ Y_t^{X_0} \in (0, x_{\ell,1}[w]] \cup [x_{\ell,2}[w], \infty)\} \tag{4.8}$$

for the problem in (4.5). A direct verification together with the chain of equalities $\mathrm{sgn}\{(\mathcal{A}_0 - (r + \lambda\gamma))(G_\ell w)(x)\} = \mathrm{sgn}\{(\mathcal{A}_0 - (r + \lambda\gamma))(h - \lambda\gamma(Hw))(x)\} = \mathrm{sgn}\{(Lw)''(F(x))\} < 0$ for every $x \in (\ell, x_{\ell,1}[w]) \cup (x_{\ell,2}[w], \infty)$ from Dayanik and Karatzas [11, page 192] prove the versions of Propositions 7 and 8 and Theorem 9 for the second problem obtained after G, H, J are replaced with G_ℓ, H_ℓ, J_ℓ and all functions are restricted to $[\ell, \infty)$. By the next theorem, optimal stopping time for asset manager's second problem is of the form $\widetilde{\tau}_\ell[w] \triangleq \inf\{t \geq 0;\ X_t \in (0, x_{\ell,1}[w]] \cup [x_{\ell,2}[w], \infty)\}$.

Theorem 15. *For every* $x \in \mathbb{R}_+$, *we have* $V_\ell(x) = v_{\ell,\infty}(x) = \mathbb{E}_x^\gamma\big[e^{-r\widetilde{\tau}_\ell[v_{\ell,\infty}]}h(X_{\widetilde{\tau}_\ell[v_{\ell,\infty}]})\big]$, *and* $\widetilde{\tau}_\ell[v_{\ell,\infty}]$ *is an optimal stopping time for (4.2).*

The proof is similar to that of Theorem 10, and Propositions 11 and 12 and Corollary 13 hold with obvious changes. We expect that the value of the limited protection at level ℓ to increase as ℓ increases. We also expect that the asset manager quits early as the protection limit ℓ increases to L. Those expectations are validated by means of the next lemma.

Lemma 16. *Let* $w : \mathbb{R}_+ \mapsto \mathbb{R}$ *be as in Assumption 5. Suppose that* $0 < \ell < u < L$. *Then*

(i) $(M_\ell w)(\cdot) \geq (M_u w)(\cdot)$ *on* $\mathbb{R}_+$, *(ii)* $0 < \zeta_{\ell,1}[w] < \zeta_{u,1}[w] < F(L) < \zeta_{u,2}[w] < \zeta_{\ell,2}[w] < \infty$,

(iii) $(J_\ell w)(\cdot) \geq (J_u w)(\cdot)$ *on* $\mathbb{R}_+$, *(iv)* $0 < x_{\ell,1}[w] < x_{u,1}[w] < L < x_{u,2}[w] < x_{\ell,2}[w] < \infty$.

Recall that $(M_\ell w)(\cdot)$ and $(M_u w)(\cdot)$ coincide, respectively, on $(0, F(\ell)]$ and $(0, F(u)]$ with $(Lw)(\cdot)$ and on $(F(\ell), \infty)$ and $(F(u), \infty)$ with the smallest nonnegative concave majorants of $(Lw)(\cdot)$, respectively, over $(F(\ell), \infty)$ and $(F(u), \infty)$. Therefore, *(i)* and *(ii)* of Lemma 16 immediately follow; see the picture on the right in Figure 2. Finally, *(iii)* and *(iv)* follow from *(i)* and *(ii)* by the relation (4.4): $(J_\ell w)(x) = \lambda\gamma(Hw)(x) + (G_\ell w)(x) = \lambda\gamma(Hw)(x) + \varphi(x)(M_\ell w)(F(x))$ for every x; $x_{\ell,1}[w] = F^{-1}(\zeta_{\ell,1}[w])$, $x_{\ell,2}[w] = F^{-1}(\zeta_{\ell,2}[w])$, and that $F(\cdot)$ is strictly increasing.

Proposition 17 shows that demanding higher portfolio insurance or limiting more severely the downward risks or losses also limits the upward potential and reduces the total value of the portfolio.

Proposition 17. *For every* $0 < \ell < u < L$, *(i)* $v_{\ell,n}(x) \geq v_{u,n}(x)$ *for all* $0 \leq n \leq \infty$, *(ii)* $U_\ell(x) \geq U_u(x)$ *for every* $x \in \mathbb{R}_+$, *and (iii)* $0 < x_{\ell,1}[v_{\ell,n}] \leq x_{u,1}[v_{u,n}] < L < x_{u,2}[v_{u,n}] \leq x_{\ell,2}[v_{\ell,n}] < \infty$.

Proof. Note first that $v_{\ell,0}(x) = h(x) = v_{u,0}(x)$ for every $x \in \mathbb{R}_+$. Suppose that $v_{\ell,n}(\cdot) \geq v_{u,n}(\cdot)$ for some $n \geq 0$. Then by the monotonicity and Lemma 16 *(iii)*, $v_{\ell,n+1}(\cdot) = (J_\ell v_{\ell,n})(\cdot) \geq (J_\ell v_{u,n})(\cdot) \geq$

$(J_u v_{u,n})(\cdot) = v_{u,n+1}(\cdot)$. Therefore, for every $n \geq 0$ $v_{\ell,n}(\cdot) \geq v_{u,n}(\cdot)$, and $v_{\ell,\infty}(\cdot) = \lim_{n\to\infty} v_{\ell,n}(\cdot) \geq \lim_{n\to\infty} v_{u,n}(\cdot) = v_{u,\infty}(\cdot)$, which proves (i). By (4.2), $U_\ell(x) = x - \frac{cL}{r} + v_{\ell,\infty}(x) \geq x - \frac{cL}{r} + v_{u,\infty}(x) = U_u(x)$ for every $x > 0$, and (ii) follows. Finally, (4.7) and (i) imply $(0, x_{\ell,1}[v_{\ell,\infty}]] \cup [x_{\ell,1}[v_{\ell,\infty}], \infty) = \{x > 0;\ v_{\ell,\infty}(x) \leq h(x)\} \subseteq \{x > 0;\ v_{u,\infty}(x) \leq h(x)\} = (0, x_{u,1}[v_{u,\infty}]] \cup [x_{u,1}[v_{u,\infty}], \infty)$. Hence, $0 < x_{\ell,1}[v_{\ell,\infty}] \leq x_{u,1}[v_{u,\infty}] < L < x_{u,2}[v_{u,\infty}] \leq x_{\ell,2}[v_{\ell,\infty}] < \infty$. Similarly, $(0, x_{\ell,1}[v_{\ell,n}]] \cup [x_{\ell,1}[v_{\ell,n}], \infty) = \{x > 0;\ v_{\ell,n+1}(x) \leq h(x)\} \subseteq \{x > 0;\ v_{u,n+1}(x) \leq h(x)\} = (0, x_{u,1}[v_{u,n}]] \cup [x_{u,1}[v_{u,n}], \infty)$, which implies $0 < x_{\ell,1}[v_{\ell,n}] \leq x_{u,1}[v_{u,n}] < L < x_{u,2}[v_{u,n}] \leq x_{\ell,2}[v_{\ell,n}] < \infty$ for every finite $n \geq 0$. □

5. Numerical illustration

For illustration, we take $L = 1$, $\sigma = 0.275$, $r = 0.03$, $c = 0.05$, $\delta = 0.08$, $\lambda\gamma = 0.01$, $y_0 = 0.03$. Observe that $0 < r < c < \delta$. We obtain $\alpha_0 = -0.3910$ and $\alpha_1 = 2.7054$. We implemented the successive approximations of Sections 3 and 4 in `R` in order to use readily available routines to calculate the smallest nonnegative concave majorants of functions. We have used `gcmlcm` function from the R package `fdrtool` developed by Korbinian Strimmer for that purpose. The approximation functions `approxfun` and `splinefun` were also useful to compactly represent the functions we evaluated on appropriate grids placed on state space and their F-transformations. By trial-and-error, we find out that optimal continuation region lies strictly inside $[0, 10L]$. Because $F(L)$ turns out to be significantly smaller than the upper bound $10L$, for the accuracy of the results it proved useful to put a grid on the interval $[0, F(L)]$ one hundred times finer than the grid put on $[F(L), F(10L)]$.

In the implementation of the successive approximations of Sections 3 and 4, we decided to stop the iterations as soon as the maximum absolute difference between the last two approximations over the grid placed on $[0, 10L]$ is less than 0.01. We obtain a good approximation for the first problem after three iterations with the maximum absolute difference $\|v_3 - v_2\| \approx 0.0011$ and returns $v_3(\cdot)$, $(0, x_1[v_2]] \cup [x_2[v_2], \infty) = (0, 0.3874] \cup [4.7968, \infty)$, and $\widetilde{\tau}[v_3] = \inf\{t \geq 0;\ X_t \notin (0, 0.3874] \cup [4.7968, \infty)\}$ as the approximate value function, approximate stopping region, and nearly optimal stopping rule for (3.6), respectively. The bound of Corollary 13 also guarantees that $\|V(\cdot) - v_3(\cdot)\| \leq \frac{cL}{r}(\frac{\lambda\gamma}{r+\lambda\gamma})^3 = 0.026$. The leftmost picture in Figure 3 suggests that the algorithm actually converges faster than what this upper bound implies. The second and third pictures illustrate how the solution of each auxiliary problem is found by constructing the smallest nonnegative concave majorants M of the transformations with operator L. The insets give closer look over the small interval $[0, F(L)]$ at the same pictures which are otherwise harder to identify. The first three pictures in Figure 3 are consistent with the general form sketched in Figure 1.

The last three pictures in Figure 3 similarly illustrate the solution of the second problem of the asset manager when the investors hold a limited protection of their assets with lower bound $\ell = 0.69$ on the market value of the asset manager's portfolio. Because $x_1[v_\infty] \approx x_1[v_2] = 0.3874 < \ell < 4.7968 = x_2[v_2] \approx x_2[v_\infty]$, the unconstrained solution of Problem 1 (corresponding to $\ell = 0$) is not any more optimal. Therefore, we calculate the successive approximations of Section 4, which converge in two iterations because $\|v_{\ell,2} - v_{\ell,1}\| \approx 0.0063 < 1/100$. Hence, $v_{\ell,2}(\cdot)$, $(0, x_{\ell,1}[v_{\ell,1}]] \cup$

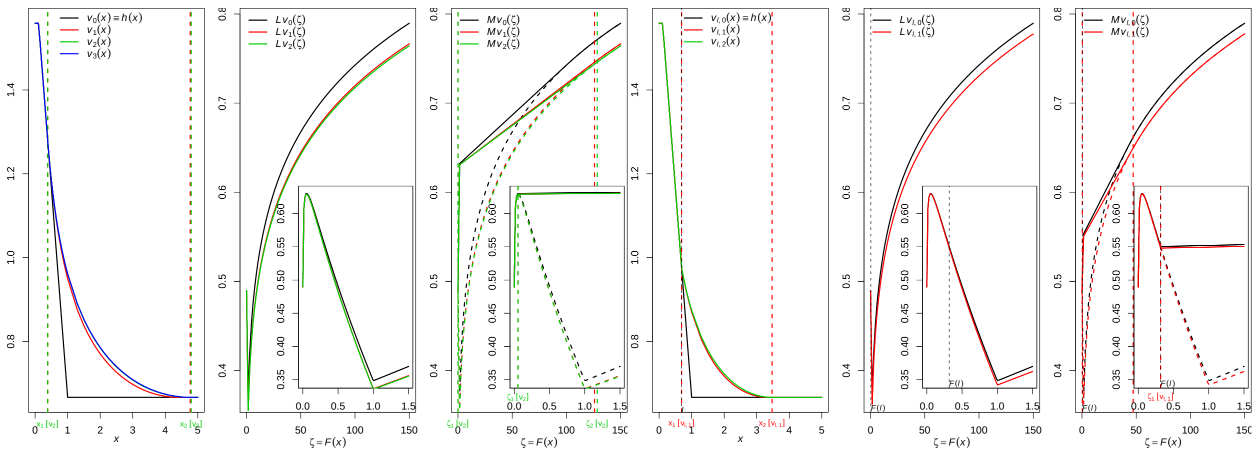

FIGURE 3. Numerical illustrations of the solutions of the auxiliary optimal stopping problems (3.6) on the left and (4.2) on the right in the first and second problems (with $\ell = 0.69$), respectively.

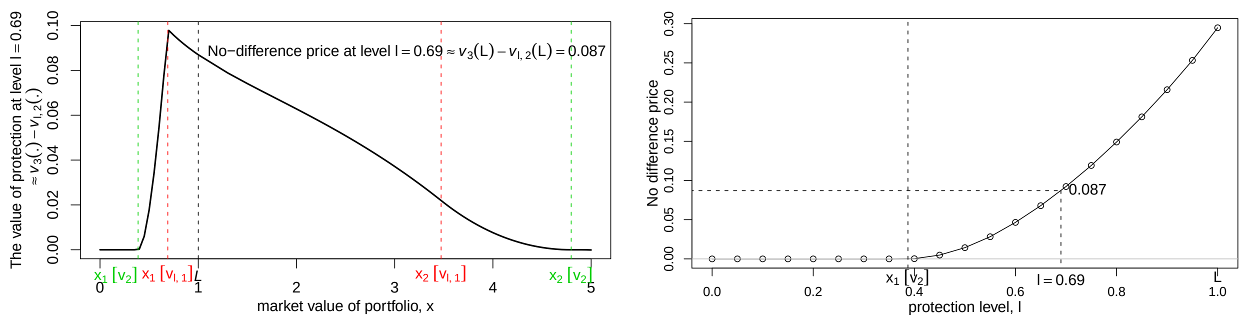

FIGURE 4. On the left, the value of the limited protection at level $\ell = 0.69$ as the market value of portfolio changes, and on the right, no-difference prices of the protections for different protection limits.

$[x_{\ell,2}[v_{\ell,1}], \infty) = (0, 0.69] \cup [3.4724, \infty)$, and $\widetilde{\tau}_\ell[v_{\ell,1}] = \inf\{t \geq 0;\ X_t \notin (0, 0.69] \cup [3.4724, \infty)\}$ are approximate value function and stopping region, and nearly optimal stopping rule for (4.2).

Observe that the stopping region of Problem 2 contains the stopping region of Problem 1: $(0, x_{\ell,1}[v_{\ell,1}]] \cup [x_{\ell,2}[v_{\ell,1}], \infty) = (0, 0.69] \cup [3.4724, \infty) \supset (0, x_1[v_2]] \cup [x_2[v_2], \infty) = (0, 0.3874] \cup [4.7968, \infty)$. Thus, asset manager stops early in the presence of portfolio protection at level $\ell = 0.69$. Because $U(x) \approx x - \frac{cL}{r} + v_2(x)$ and $U_\ell(x) \approx x - \frac{cL}{r} + v_{\ell,1}(x)$ are approximately the value functions of Problems 1 and 2, the value of the limited protection at level ℓ when stock price is x equals $U(x) - U(\ell)(x) \approx v_3(x) - v_{\ell,2}(x)$, which is plotted on the left in Figure 4. Therefore, the no-difference price of this protection at the initiation of the contract equals $U(L) - U(\ell)(L) \approx v_3(L) - v_{\ell,2}(L) = 0.087$. The plot on the right in Figure 4 shows the no-difference prices of the protection at levels ℓ changing between 0 and $L = 1$. The protection has no value at the protection levels less than or equal to $x_1[v_\infty] \approx x_1[v_2]$, because the optimal policy, even in the absence of protection clause, instructs the asset manager to quit as soon as the market value of the portfolio goes below $x_1[v_\infty] \approx x_1[v_2]$.

Let us finish with a final remark about the role of L. Let us replace $U(\cdot)$ in (2.2) with $U_L(\cdot)$ to emphasize its dependence on $L > 0$. Then

$$\begin{aligned} U_L(x) &= \sup_{\tau\in\mathcal{S}} \mathbb{E}_x^\gamma\left[e^{-r\tau}(X_\tau - L)^+ + \int_0^\tau e^{-rt}(\delta X_t - cL)\mathrm{d}t\right] \\ &= \sup_{\tau\in\mathcal{S}} L\,\mathbb{E}_x^\gamma\left[e^{-r\tau}(X_\tau/L - 1)^+ + \int_0^\tau e^{-rt}(\delta X_t/L - c)\mathrm{d}t\right] \\ &= \sup_{\tau\in\mathcal{S}} L\,\mathbb{E}_{x/L}^\gamma\left[e^{-r\tau}(X_\tau - 1)^+ + \int_0^\tau e^{-rt}(\delta X_t - c)\mathrm{d}t\right] = L\,U_1(x/L) \quad \text{for every } x > 0. \end{aligned}$$

Therefore, we can in fact choose $L = 1$ in (2.2) without loss of generality and solve it for $U_1(\cdot)$ and obtain the solutions for all other $L > 0$ values by the transformation $U_L(x) = L\,U_1(x/L)$ for every $x > 0$.

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