SEQUENTIAL TESTING OF SIMPLE HYPOTHESES ABOUT COMPOUND POISSON PROCESSES

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Abstract. One of two simple hypotheses is correct about the unknown arrival rate and jump distribution of a compound Poisson process. We start observing the process, and the problem is to decide on the correct hypothesis as soon as possible and with the smallest probability of wrong decision. We find a Bayes-optimal sequential decision rule and describe completely how to calculate its parameters without any restrictions on the arrival rate and the jump distribution.

1. Introduction

Let $N = \{N_t;\, t \geq 0\}$ be a simple Poisson process with arrival rate λ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Independent of the process N, let $Y_1, Y_2, \ldots$ be i.i.d. $\mathbb{R}^d$-valued random variables with some common distribution $\nu(\cdot)$. The pair $(\lambda, \nu(\cdot))$ is the *unknown* characteristic of the compound Poisson process

$$X_t = X_0 + \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0. \tag{1.1}$$

Suppose that exactly one of two simple hypotheses

$$H_0 : (\lambda, \nu(\cdot)) = (\lambda_0, \nu_0(\cdot)) \qquad \text{and} \qquad H_1 : (\lambda, \nu(\cdot)) = (\lambda_1, \nu_1(\cdot)) \tag{1.2}$$

is correct, and the alternatives $(\lambda_0, \nu_0(\cdot))$ and $(\lambda_1, \nu_1(\cdot))$ are known. At time $t = 0$, we know only that the hypotheses H_0 and H_1 are correct with prior probabilities $1 - \pi$ and $\pi \in [0, 1)$, respectively, and start observing the process $X = \{X_t;\ t \geq 0\}$. Our objective is to decide *as soon as possible* between the null hypothesis H_0 and its alternative H_1 with the *smallest* probability of wrong decision.

Any admissible decision rule is a pair (τ, d) of a stopping time $\tau : \Omega \to [0, \infty]$ of the observation process X and a random variable $d : \Omega \mapsto \{0, 1\}$ whose value is determined

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completely by the history $\{X_{t\wedge\tau};\ t\ge 0\}$ of the process X at time τ. On the event $\{\tau<\infty\}$, we select at time τ the null hypothesis H_0 if $d=0$, and the alternative hypothesis H_1 otherwise.

A wrong decision is made if either $d=1$ and H_0 is correct (Type I error), or $d=0$ and H_1 is correct (Type II error). The costs of Type I and Type II errors are some positive constants b and a, respectively.

For every admissible decision rule (τ,d) we define the *Bayes risk* as

$$R_{\tau,d}(\pi)=\mathbb{E}\left[\tau+\left(a\cdot 1_{\{d=0,\ H_1\text{ is correct}\}}+b\cdot 1_{\{d=1,\ H_0\text{ is correct}\}}\right)\cdot 1_{\{\tau<\infty\}}\right],\quad \pi\in[0,1). \tag{1.3}$$

Our problem is to calculate the *minimum Bayes risk*

$$U(\pi)\triangleq\inf_{(\tau,d)}R_{\tau,d}(\pi),\qquad \pi\in[0,1) \tag{1.4}$$

over all admissible decision rules and to find (if it exists) an admissible decision rule which attains the infimum for every $\pi\in[0,1)$. If the Bayes risk $R_{\tau,d}(\cdot)$ in (1.3) is the minimum, then the rule (τ,d) is *Bayes-optimal*: it solves optimally the trade-off between the expected length of observation before a decision is made and the probabilities of making a wrong decision.

Special problems of sequential testing for compound Poisson processes have been studied by Peskir and Shiryaev (2000) and Gapeev (2002). Peskir and Shiryaev (2000) solved the problem in (1.3, 1.4) when the Poisson process X is *simple*. Equivalently, the mark distribution $\nu(\cdot)$ is known (i.e., $\nu_0(\cdot)\equiv\nu_1(\cdot)$), and the objective is to find an admissible decision rule (τ,d) with minimum Bayes risk $R_{\tau,d}(\cdot)$ in order to decide between the hypotheses $H_0:\lambda=\lambda_0$ and $H_1:\lambda=\lambda_1$; compare with (1.2).

For the first time, Gapeev (2002) studied sequential testing of unknown arrival rate λ *and* mark distribution $\nu(\cdot)$ as in (1.2), but assumed that they are very special: the distribution $\nu(\cdot)$ is exponential on $\mathbb{R}_+$, and the expected value $\int_0^\infty y\,\nu(dy)$ of the marks is the same as their arrival rate λ.

The contribution of this paper is the complete Bayes solution of the sequential testing problem of simple hypotheses in (1.2) for a general compound Poisson process. The problem is non-trivial if the distributions $\nu_0(\cdot)$ and $\nu_1(\cdot)$ are equivalent. In this case, an optimal admissible decision rule $(U_0,d(U_0))$ is described in terms of the likelihood ratio process

$\Phi = \{\Phi_t;\ t \geq 0\}$ of (2.4, 2.5): for some suitable constants $0 < \xi_0 < b/a < \xi_1 < \infty$, if the rule

$$d(U_0) \triangleq \begin{cases} 0 \text{ (choose the null hypothesis } H_0), & \text{if} \quad \Phi_{U_0} \leq b/a \\ 1 \text{ (choose the alternative hypothesis } H_1), & \text{if} \quad \Phi_{U_0} > b/a \end{cases}$$

is applied at the first time

$$U_0 \triangleq \inf\{t \geq 0 :\ \Phi_t \notin (\xi_0, \xi_1)\}$$

that the process Φ exits the interval (ξ_0, ξ_1), then the corresponding Bayes risk $R_{(U_0, d(U_0))}(\cdot)$ is the smallest in (1.3, 1.4) among all admissible decision rules. We describe an accurate numerical algorithm in order to calculate the critical thresholds ξ_0, ξ_1, and the minimum Bayes risk $U(\cdot)$.

The process Φ jumps at the arrival times of the observation process X and evolves deterministically between them. It is a piecewise-deterministic Markov process and can be updated recursively. This special structure of the process is crucial for our analytical and numerical results.

The decision rule $(U_0, d(U_0))$ is the well-known *Sequential Probability Ratio Test (SPRT)*. It is easy to check that this test has the smallest expected observation time under both hypotheses among all admissible decision rules whose Type I and II error probabilities are not greater than those of the SPRT.

In fact, the SPRT is known to be optimal for the *fixed error probability formulation* of a wide class of sequential testing problems of simple hypotheses, including (1.2) for a compound Poisson process. In this formulation of the compound Poisson case, there is, however, no procedure to calculate the boundaries ξ_0 and ξ_1 of the optimal SPRT with pre-determined Type I and II error probabilities. We are hoping to address this problem in the future by using the numerical solution method of this paper for the Bayesian formulation.

The optimality of the SPRT was proved by Wald and Wolfowitz (1948) for the fixed error probability formulation of testing two simple hypotheses about unknown common distribution of i.i.d. random variables, which are observed sequentially. Shiryaev (1978, Chapter 4) proved that the SPRT is optimal for both Bayes and fixed error probability formulations of testing two simple hypotheses about the unknown drift of a linear Brownian motion. Irle and Schmitz (1984) showed that the SPRT is optimal for fixed error probability formulation for a wider class of continuous-time processes. Recently, Peskir and Shiryaev (2000) showed the optimality of the SPRT for both formulations of sequential testing of two simple hypotheses about unknown arrival rate of a simple Poisson process. See, also, the forthcoming book by Peskir and Shiryaev (2006) for an up-to-date presentation of major techniques and important results.

In Section 2, we describe the problem and reduce it to an optimal stopping problem for a Markov process. In Section 3, accurate successive approximations of latter problem's value function are obtained. They are used in Section 4 to identify the structure of an optimal decision rule and an efficient numerical method to calculate its parameters. Results are illustrated on several old and new examples in Section 5. Finally, we investigate in Section 6 the analytical properties of the solution. Long derivations are deferred to the Appendix.

2. Model and problem description

In this section we construct a probability model of the random elements described in the introduction by means of a reference probability measure.

2.1. **Model:** Let $(\Omega, \mathcal{F}, \mathbb{P}_0)$ be a probability space on which the process X of (1.1) is a compound Poisson process with arrival rate λ_0 and jump distribution $\nu_0(\cdot)$ ($\nu_0(\{0\}) = 0$). Moreover, let Θ be an independent random variable with the distribution

$$\mathbb{P}_0\{\Theta = 1\} = \pi \qquad \text{and} \qquad \mathbb{P}_0\{\Theta = 0\} = 1 - \pi. \tag{2.1}$$

Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration of X enlarged with $\mathbb{P}_0$-null sets and $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$, $\mathcal{G}_t \triangleq \mathcal{F}_t \vee \sigma(\Theta)$ be its augmentation by the events in $\sigma(\Theta)$. We replace $\mathcal{F}$ with $\vee_{t\geq 0}\mathcal{G}_t$.

Let $\lambda_1 \geq \lambda_0$ be a constant and $\nu_1(\cdot)$ be a $\nu_0(\cdot)$-equivalent probability distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ with the Radon-Nikodym derivative

$$f(y) \triangleq \left.\frac{d\nu_1}{d\nu_0}\right|_{\mathscr{B}(\mathbb{R}^d)}(y), \quad y \in \mathbb{R}^d. \tag{2.2}$$

We define a new probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ by specifying it locally in terms of the Radon-Nikodym derivatives

$$\left.\frac{d\mathbb{P}}{d\mathbb{P}_0}\right|_{\mathcal{G}_t} = Z_t \triangleq 1_{\{\Theta=0\}} + 1_{\{\Theta=1\}} \cdot e^{-(\lambda_1-\lambda_0)t} \prod_{i=1}^{N_t} \left[\frac{\lambda_1}{\lambda_0} f(Y_i)\right], \qquad 0 \leq t < \infty. \tag{2.3}$$

Under the new probability measure $\mathbb{P}$, the $\mathbb{G}$-adapted marked point process X is a compound Poisson process with arrival rate $(1-\Theta)\lambda_0 + \Theta\lambda_1$ and mark distribution $(1-\Theta)\nu_0(\cdot) + \Theta\nu_1(\cdot)$ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$: if $\Theta = 0$, then the probability measures $\mathbb{P}$ and $\mathbb{P}_0$ coincide on $(\Omega, \mathcal{F})$; if $\Theta = 1$, then Z_t in (2.3) coincides with the likelihood ratio

$$L_t \triangleq e^{-(\lambda_1-\lambda_0)t} \prod_{i=1}^{N_t} \left[\frac{\lambda_1}{\lambda_0} f(Y_i)\right], \qquad 0 \leq t < \infty \tag{2.4}$$

of the finite-dimensional distributions of two compound Poisson processes with characteristics $(\lambda_1, \nu_1(\cdot))$ and $(\lambda_0, \nu_0(\cdot))$, respectively; see also Appendix A.1.

Finally, $Z_0 \equiv 1$ and $\mathbb{P} \equiv \mathbb{P}_0$ on $\mathcal{G}_0$. Therefore, the $\mathcal{G}_0$-measurable random variable Θ has the same distribution under $\mathbb{P}$ and $\mathbb{P}_0$. Hence, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we obtain the same setup as described in the introduction.

2.2. **Problem description.** In the remainder, we shall work with the explicit model constructed above. The main result of this section describes below an optimal decision rule at every stopping time of the process X. Therefore, sequential hypothesis testing problem reduces to an optimal stopping problem. In the following sections we solve the optimal stopping problem and identify an optimal time to stop and decide between two hypotheses.

Let $\Phi = \{\Phi_t;\ t \geq 0\}$ be the same as the likelihood ratio process $L = \{L_t;\ t \geq 0\}$ in (2.4) starting from an arbitrary fixed point $\Phi_0 \geq 0$; namely,

$$\Phi_t \triangleq \Phi_0 \cdot L_t, \qquad t \geq 0. \tag{2.5}$$

Any *admissible decision rule* is a pair (τ, d) of a stopping time $\tau : \Omega \mapsto [0, \infty]$ of the filtration $\mathbb{F}$ (i.e., $\tau \in \mathbb{F}$) and a random variable $d : \Omega \mapsto \{0, 1\}$ measurable with respect to the σ-algebra $\mathcal{F}_\tau = \sigma\{X_{t\wedge\tau};\ t \geq 0\}$ (i.e., $d \in \mathcal{F}_\tau$).

2.1. **Proposition.** *For every $\pi \in [0, 1)$ and admissible decision rule (τ, d), the Bayes risk in (1.3) can be written as*

$$R_{\tau,d}(\pi) = b(1-\pi)\mathbb{P}_0\{\tau < \infty\} + (1-\pi)\mathbb{E}_0^{\frac{\pi}{1-\pi}}\left[\int_0^\tau (1+\Phi_t)dt + (a\Phi_\tau - b)1_{\{d=0,\tau<\infty\}}\right], \tag{2.6}$$

where the expectation $\mathbb{E}_0^\phi$ is taken with respect to the probability measure $\mathbb{P}_0^\phi$, which is the same as $\mathbb{P}_0$ such that $\mathbb{P}_0\{\Phi_0 = \phi\} = 1$. If we define

$$d(\tau) \triangleq \begin{Bmatrix} 0, & \text{if} \quad \Phi_\tau \leq b/a \\ 1, & \text{if} \quad \Phi_\tau > b/a \end{Bmatrix} \cdot 1_{\{\tau<\infty\}} \in \mathcal{F}_\tau, \tag{2.7}$$

then $(\tau, d(\tau))$ is admissible, and $R_{\tau,d}(\pi) \geq R_{\tau,d(\tau)}(\pi)$ for every $\pi \in [0, 1)$. The minimum Bayes risk in (2.7) equals

$$U(\pi) = \inf_{\tau\in\mathbb{F}} R_{\tau,d(\tau)}(\pi) = b(1-\pi) + (1-\pi)\cdot V\left(\frac{\pi}{1-\pi}\right), \qquad \pi \in [0, 1) \tag{2.8}$$

in terms of the function ($x^- \triangleq \max\{0, -x\}$)

$$V(\phi) \triangleq \inf_{\tau\in\mathbb{F}} \mathbb{E}_0^\phi\left[\int_0^\tau \Big(1+\Phi_t\Big)dt - (a\Phi_\tau - b)^- 1_{\{\tau<\infty\}}\right], \qquad \phi \in \mathbb{R}_+. \tag{2.9}$$

Proposition 2.1 implies that the minimum Bayes risk $U(\cdot)$ can be found as in (2.8) by calculating first the value function $V(\cdot)$ of the optimal stopping problem in (2.9). If that problem admits an optimal stopping time τ^*, then the admissible decision rule $(\tau^*, d(\tau^*))$ is Bayes-optimal for (1.4): observe the process $\Phi = \{\Phi_t; t \geq 0\}$ until time τ^* and then stop; on the event $\{\tau^* < \infty\}$, select the null hypothesis H_0 (respectively, its alternative H_1) if $\Phi_{\tau^*} \leq b/a$ (respectively, $\Phi_{\tau^*} > b/a$).

The underlying process Φ of the optimal stopping problem in (2.9) can be expressed as (see Appendix A.2)

$$\left\{ \begin{array}{c} \Phi_t = x\left(t - \sigma_{n-1}, \Phi_{\sigma_{n-1}}\right), \quad t \in [\sigma_{n-1}, \sigma_n) \\ \Phi_{\sigma_n} = \dfrac{\lambda_1}{\lambda_0} f(Y_n) \Phi_{\sigma_n -} \end{array} \right\}, \quad n \geq 1 \tag{2.10}$$

in terms of the deterministic function

$$x(t, \phi) = \phi \cdot e^{-(\lambda_1 - \lambda_0)t}, \qquad (t, \phi) \in \mathbb{R} \times \mathbb{R}, \tag{2.11}$$

the Radon-Nikodym derivative $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ in (2.2) of the distribution $\nu_1(\cdot)$ with respect to $\nu_0(\cdot)$, and the arrival times of the point process X in (1.1)

$$\sigma_n \triangleq \inf\{t > \sigma_{n-1} : X_t \neq X_{t-}\}, \quad n \geq 1 \qquad (\sigma_0 \equiv 0). \tag{2.12}$$

The process Φ is a piecewise-deterministic Markov process with random jump magnitudes. Between successive jumps of the process X, every sample-path of Φ decreases asymptotically to 0 along the curves $t \mapsto x(t, \cdot)$ of (2.11) if $\lambda_1 > \lambda_0$, and stays constant if $\lambda_1 = \lambda_0$. At every jump time σ_n, it is adjusted instantaneously by the proportion $(\lambda_1/\lambda_0) f(Y_n)$ up or down. See Figure 1.

In Appendix A.3, the infinitesimal generator of the Markov process Φ is shown to coincide on the collection of continuously differentiable functions H with the integro-differential operator

$$\mathcal{A}H(\phi) \triangleq -(\lambda_1 - \lambda_0)\, \phi H'(\phi) + \lambda_0 \int_{\mathbb{R}^d} \left[H\left(\frac{\lambda_1}{\lambda_0} f(y) \phi \right) - H(\phi) \right] \nu_0(dy). \tag{2.13}$$

The dynamic programming principle suggests that the value function $V(\cdot)$ of the optimal stopping problem in (2.9) must satisfy the variational inequalities

$$\min\left\{ \mathcal{A}v(\phi) + 1 + \phi,\, (a\phi - b)^- - v(\phi) \right\} = 0 \tag{2.14}$$

under suitable conditions and may be identified explicitly by solving (2.14). However, (2.14) is not easy to analyze analytically due to the integro-differential operator $\mathcal{A}$. Instead, we use successive approximations whose details are deferred to the next section. This method,

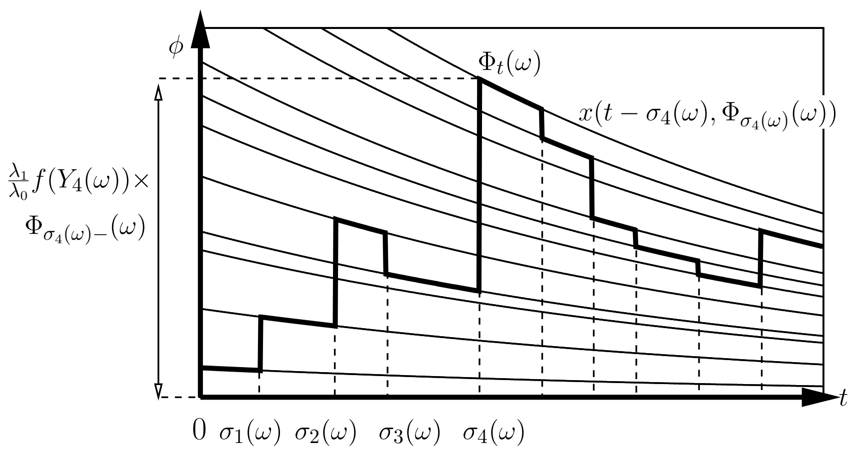

FIGURE 1. A sample-path of the process Φ in (2.5, 2.10) when $\lambda_1 > \lambda_0$. The deterministic function $x : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is given by (2.11), and the function $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ is the density in (2.2) of the jump distribution $\nu_1(\cdot)$ under H_1 with respect to the jump distribution $\nu_0(\cdot)$ under H_0. The process Φ rides on the curves $t \mapsto x(t, \phi)$, $\phi \in \mathbb{R}_+$. At every arrival time $\sigma_1, \sigma_2, \ldots$ of the observation process X in (1.1), the process Φ jumps onto a new curve; the jump size depends on the mark size $Y_1, Y_2, \ldots$ of the arrival. If $\lambda_1 = \lambda_0$, then the curves $t \mapsto x(t, \phi)$, $\phi \in \mathbb{R}$ are flat.

being easy to implement numerically, is very suitable for piecewise-deterministic processes. In addition, as we will see in later sections, it allows us to show that the value function $V(\cdot)$ of (2.9) is actually the unique solution of (2.14) under suitable conditions. A similar approach has been taken by Bayraktar, Dayanik, and Karatzas (2006) and Dayanik and Sezer (2006) in order to solve optimal stopping problems arising from sequential change detection problems for Poisson processes. However, unlike in the aforementioned papers, the optimal stopping problem of this paper involves a nonzero terminal penalty and no discount factor, both of which make the current analysis significantly harder and more interesting.

3. Successive approximations

Let us denote the running and terminal cost functions of the problem in (2.9) by

$$g(\phi) \triangleq 1 + \phi \qquad \text{and} \qquad h(\phi) \triangleq -(a\phi - b)^-, \tag{3.1}$$

respectively, and introduce the family of optimal stopping problems

$$V_n(\phi) \triangleq \inf_{\tau \in \mathbb{F}} \mathbb{E}_0 \left[\int_0^{\tau\wedge\sigma_n} g(\Phi_t)\, dt + h(\Phi_{\tau\wedge\sigma_n}) \right], \qquad n \geq 1, \tag{3.2}$$

obtained from the original problem in (2.9) by stopping the process Φ at the nth jump time. Since the sequence of jump times $\{\sigma_n\}_{n\geq 1}$ is increasing, the sequence $\{V_n(\cdot)\}_{n\geq 1}$ is decreasing, and $\lim_{n\to\infty} V_n$ exists. Since $g(\cdot) \geq 1$ and $0 \geq h(\cdot) \geq -b$, we also have $-b \leq V(\cdot) \leq V_n(\cdot) \leq h(\cdot) \leq 0$. Therefore, $V(\cdot)$ and $V_n(\cdot)$, $n \geq 1$ are bounded.

3.1. **Proposition.** *As $n \to \infty$, we have $V_n(\cdot) \searrow V(\cdot)$ on $\mathbb{R}_+$.*

Later in Section 4 (see Proposition 4.4) we shall show that the convergence $V_n(\cdot) \searrow V(\cdot)$ is uniform on $\mathbb{R}_+$. To calculate the functions $V_n(\cdot)$, $n \geq 1$ successively, we define the following operators acting on bounded functions $w : \mathbb{R}_+ \to \mathbb{R}$:

$$Jw(t,\phi) \triangleq \mathbb{E}_0^{\phi}\left[\int_0^{t\wedge\sigma_1} g(\Phi_u)du + 1_{\{t<\sigma_1\}}h(\Phi_t) + 1_{\{t\geq\sigma_1\}}w(\Phi_{\sigma_1})\right], \quad t\in[0,\infty], \tag{3.3}$$

$$J_t w(\phi) \triangleq \inf_{u\in[t,\infty]} Jw(u,\phi), \qquad t\in[0,\infty]. \tag{3.4}$$

Since the first arrival time σ_1 of the process X has exponential distribution with rate λ_0 under $\mathbb{P}_0$, the explicit dynamics of Φ in (2.10) gives

$$Jw(t,\phi) = \int_0^t e^{-\lambda_0 u}\Big[g + \lambda_0\cdot Sw\Big]\big(x(u,\phi)\big)du + e^{-\lambda_0 t}h\big(x(t,\phi)\big), \tag{3.5}$$

where $x(\cdot,\cdot)$ is the same deterministic function in (2.11), and the operator S is defined as

$$Sw(\phi) \triangleq \int_{y\in\mathbb{R}^d} \nu_0(dy)\, w\left(\frac{\lambda_1}{\lambda_0}f(y)\phi\right), \qquad \phi\in\mathbb{R}_+. \tag{3.6}$$

Moreover, using the special decomposition of the stopping times of the jump processes (see Lemma 3.6 below), one can show that

$$J_0 w(\phi) = \inf_{\tau\in\mathbb{F}} \mathbb{E}_0^{\phi}\left[\int_0^{\tau\wedge\sigma_1} g(\Phi_u)du + 1_{\{\tau<\sigma_1\}}h(\Phi_\tau) + 1_{\{\tau\geq\sigma_1\}}w(\Phi_{\sigma_1})\right]. \tag{3.7}$$

Let us define successively a sequence of functions $\{v_n\}_{n\in\mathbb{N}}$ by

$$v_0 \triangleq h \qquad \text{and} \qquad v_{n+1} \triangleq J_0 v_n, \quad n\geq 0. \tag{3.8}$$

We shall show by Proposition 3.5 that the functions $v_n(\cdot)$ and $V_n(\cdot)$ are identical for every $n \geq 0$. Therefore, the sequence $\{v_n(\cdot)\}_{n\in\mathbb{N}}$ converges to $V(\cdot)$ by Proposition 3.1.

3.2. **Remark.** Using the explicit form of $x(u,\phi)$ in (2.11), it is easy to check that the integrand in (3.5) is absolutely integrable on $\mathbb{R}_+$ for every bounded $w : \mathbb{R}_+ \to \mathbb{R}$. Therefore,

$$\lim_{t\to\infty} Jw(t,\phi) = Jw(\infty,\phi) < \infty,$$

and the mapping $t \mapsto Jw(t,\phi)$ from the extended nonnegative real numbers $[0,\infty]$ into the real numbers is continuous. Therefore, the infimum $J_t w(\phi)$ in (3.4) is attained for every $t \in [0,\infty]$.

3.3. **Remark.** If $w_1(\cdot) \le w_2(\cdot)$, then $Sw_1(\cdot) \le Sw_2(\cdot)$, $Jw_1(\cdot,\cdot) \le Jw_2(\cdot,\cdot)$, and $J_0 w_1(\cdot) \le J_0 w_2(\cdot)$. If $w(\cdot)$ is concave, then $Sw(\cdot)$, $Jw(t,\cdot)$ for every $t \ge 0$, and $J_0 w(\cdot)$ are concave. Finally, if $w(\cdot)$ is bounded and $w(\cdot) \ge -b$, then $-b \le J_0 w(\cdot) \le h(\phi) \le 0$.

Proof. We shall verify the last claim only; the rest are easy to check. We always have $J_0 w(\phi) \le Jw(0,\phi) \le h(\phi) \le 0$. Suppose that $w(\cdot)$ is bounded and $w(\cdot) \ge -b$. Then $Sw(\cdot)$ is well-defined, and $Sw(\cdot) \ge -b$. Since $g(\cdot) \ge 0$ and $h(\cdot) \ge -b$, (3.4) implies $Jw(t,\phi) \ge -b$ for every $t \ge 0$. Therefore, $J_0 w(\phi) = \inf_{t\in[0,\infty]} Jw(t,\phi) \ge -b$. □

3.4. **Proposition.** *The sequence $\{v_n(\cdot)\}_{n\ge 0}$ in (3.8) is decreasing with a limit*

$$v(\phi) \triangleq \lim_{n\to\infty} v_n(\phi), \qquad \phi \in \mathbb{R}_+.$$

We have $-b \le v(\cdot) \le v_n(\cdot) \le h(\cdot) \le 0$ and $v(0) = v_n(0) = -b$ for every $n \ge 0$. Both $v(\cdot)$ and $v_n(\cdot)$, $n \ge 0$ are concave, nondecreasing, and continuous on $\mathbb{R}_+$. Their left and right derivatives are bounded on every compact subset of $\mathbb{R}_+$.

3.5. **Proposition.** *For every $n \ge 0$, we have $v_n(\cdot) = V_n(\cdot)$. For every $\varepsilon \ge 0$, let*

$$r_n^\varepsilon(\phi) \triangleq \inf\left\{s \in (0,\infty] : Jv_n\big(s,\phi\big) \le J_0 v_n(\phi) + \varepsilon\right\}, \quad n \ge 0,\ \phi \in \mathbb{R}_+,$$

$$S_1^\varepsilon \triangleq r_0^\varepsilon\big(\Phi_0\big) \wedge \sigma_1, \quad \textit{and} \quad S_{n+1}^\varepsilon(\phi) \triangleq \left\{\begin{matrix} r_n^{\varepsilon/2}\big(\Phi_0\big), & \textit{if } \sigma_1 > r_n^{\varepsilon/2}\big(\Phi_0\big) \\ \sigma_1 + S_n^{\varepsilon/2} \circ \theta_{\sigma_1}, & \textit{if } \sigma_1 \le r_n^{\varepsilon/2}\big(\Phi_0\big) \end{matrix}\right\}, \quad n \ge 1,$$

where θ_s is the shift-operator on Ω: $X_t \circ \theta_s = X_{s+t}$. Then

$$\mathbb{E}_0^\phi\left[\int_0^{S_n^\varepsilon} g\big(\Phi_t\big)dt + h(\Phi_{S_n^\varepsilon})\right] \le v_n(\phi) + \varepsilon, \quad \forall\, n \ge 1,\ \forall\, \varepsilon \ge 0. \tag{3.9}$$

Proposition 3.5 gives ε-optimal stopping rules for the problems in (3.2). Its proof in Appendix A.4 follows from the strong Markov property and the next characterization of the $\mathbb{F}$-stopping times; see Brémaud (1981, Theorem T33, p. 308), Davis (1993, Lemma A2.3, p. 261).

3.6. **Lemma.** *For every $\mathbb{F}$-stopping time τ and every $n \ge 0$, there is an $\mathcal{F}_{\sigma_n}$-measurable random variable $R_n : \Omega \mapsto [0,\infty]$ such that $\tau \wedge \sigma_{n+1} = (\sigma_n + R_n) \wedge \sigma_{n+1}$ $\mathbb{P}_0$-a.s. on $\{\tau \ge \sigma_n\}$.*

3.7. **Proposition.** *We have $v(\phi) \triangleq \lim_{n\to\infty} v_n(\phi) = V(\phi)$ for every $\phi \in \mathbb{R}_+$. Moreover, V is the largest solution of $U = J_0 U$ smaller than or equal to h.*

Proposition 3.7 hints the numerical algorithm in Figure 2 described in detail in Section 4 in order to solve the optimal stopping problem in (2.9). We continue by deriving *dynamic programming equations* satisfied by the functions $v_n(\cdot)$, $n \geq 1$ and $v(\cdot)$. These equations will be useful to establish an optimal stopping rule by Proposition 3.13 and analytical properties of the value function $V(\cdot)$ in Section 6.

3.8. **Lemma.** *For every bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}$, we have*

$$J_t w(\phi) = Jw(t,\phi) + e^{-\lambda_0 t}\left[J_0 w\big(x(t,\phi)\big) - h\big(x(t,\phi)\big)\right], \qquad t \in \mathbb{R}_+,\ \phi \in \mathbb{R}_+. \tag{3.10}$$

3.9. **Corollary.** *Let $r_n(\phi) = \inf\left\{s \in (0,\infty] : Jv_n\big(s,\phi\big) = J_0 v_n(\phi)\right\}$ be the same as $r_n^\varepsilon(\phi)$ in Proposition 3.5 with $\varepsilon = 0$. Then*

$$r_n(\phi) = \inf\left\{t > 0 : v_{n+1}\big(x(t,\phi)\big) = h\big(x(t,\phi)\big)\right\} \qquad (\inf \varnothing \equiv \infty). \tag{3.11}$$

3.10. **Remark.** For every $t \in [0, r_n(\phi)]$, we have $J_t v_n(\phi) = J_0 v_n(\phi) = v_{n+1}(\phi)$. Then substituting $w(\cdot) = v_n(\cdot)$ in (3.10) gives the *dynamic programming equation* for the family $\{v_n(\cdot)\}_{n\geq 0}$: for every $\phi \in \mathbb{R}_+$ and $n \geq 0$

$$v_{n+1}(\phi) = Jv_n(t,\phi) + e^{-\lambda_0 t}\Big[v_{n+1}(x(t,\phi)) - h\big(x(t,\phi)\big)\Big], \quad t \in [0, r_n(\phi)]. \tag{3.12}$$

3.11. **Remark.** Since $V(\cdot)$ is bounded by Propositions 3.4 and 3.7, and $V = J_0 V$ by Lemma 3.8, we obtain

$$J_t V(\phi) = JV(t,\phi) + e^{-\lambda_0 t}\Big[V\big(x(t,\phi)\big) - h\big(x(t,\phi)\big)\Big], \quad t \in \mathbb{R}_+ \tag{3.13}$$

for every $\phi \in \mathbb{R}_+$. If we define

$$r(\phi) \triangleq \inf\{t > 0 : JV(t,\phi) = J_0 V(\phi)\}, \qquad \phi \in \mathbb{R}_+, \tag{3.14}$$

then (3.13) and same arguments as in the proof of Corollary 3.9 with obvious changes give

$$r(\phi) = \inf\{t > 0 : V(x(t,\phi)) = h\big(x(t,\phi)\big)\}, \qquad \phi \in \mathbb{R}_+, \tag{3.15}$$

$$V(\phi) = JV(t,\phi) + e^{-\lambda_0 t}\Big[V(x(t,\phi)) - h\big(x(t,\phi)\big)\Big], \quad t \in [0, r(\phi)]. \tag{3.16}$$

Since $V(\cdot)$ is continuous by Propositions 3.4 and 3.7, the paths $t \mapsto V(x(t,\phi))$, $\phi \in \mathbb{R}_+$ are continuous. Because the process Φ has right-continuous sample-paths with left limits, the paths $t \mapsto V(\Phi_t) = v(\Phi_t)$ are right-continuous and have left-limits. Therefore, if

$$U_\varepsilon \triangleq \inf\{t \geq 0 : h(\Phi_t) \leq V(\Phi_t) + \varepsilon\}, \qquad \varepsilon \geq 0. \tag{3.17}$$

then $h\big(\Phi_{U_\varepsilon}\big) \leq V(\Phi_{U_\varepsilon}) + \varepsilon$ on the event $\{U_\varepsilon < \infty\}$. The next two propositions verify that the $\mathbb{F}$-stopping times U_ε, $\varepsilon \geq 0$ are ε-optimal for the problem in (2.9).

3.12. **Proposition.** *Let* $M_t \triangleq V(\Phi_t) + \int_0^t g(\Phi_s)ds$, $t \geq 0$. *For every* $n \geq 1$, $\varepsilon \geq 0$, *and* $\phi \in \mathbb{R}_+$, *we have* $V(\phi) = \mathbb{E}_0^\phi[M_0] = \mathbb{E}_0^\phi[M_{U_\varepsilon \wedge \sigma_n}]$, *i.e.,*

$$V(\phi) = \mathbb{E}_0^\phi \left[V(\Phi_{U_\varepsilon \wedge \sigma_n}) + \int_0^{U_\varepsilon \wedge \sigma_n} g(\Phi_s)ds \right]. \tag{3.18}$$

3.13. **Proposition.** *For every* $\varepsilon \geq 0$, *the stopping time* U_ε *has finite expectation under* $\mathbb{P}_0$ *and is an* ε*-optimal stopping time for the optimal stopping problem (2.9), i.e.,*

$$\mathbb{E}_0^\phi \left[\int_0^{U_\varepsilon} g(\Phi_s)ds + 1_{\{U_\varepsilon < \infty\}} h(\Phi_{U_\varepsilon}) \right] \leq V(\phi) + \varepsilon, \quad \textit{for every} \quad \phi \in \mathbb{R}_+.$$

The following results will be needed later to show that the convergence of the sequence $\{V_n(\cdot)\}_{n\geq 0}$ to $V(\cdot)$ is *uniform* on $\mathbb{R}_+$. They imply that the exit time of the process Φ in (2.5, 2.10) from every bounded interval away from the origin is finite $\mathbb{P}_0$-a.s.

3.14. **Proposition.** *Let* $\hat{\tau} \triangleq \inf\{t \geq 0;\ \Phi_t \notin (\phi_0, \phi_1)\}$ *be the exit time of the process* Φ *from the interval* (ϕ_0, ϕ_1) *for some* $0 < \phi_0 < \phi_1 < \infty$. *Then there exists an integer* $k \geq 1$ *and a constant* $p \in (0,1)$ *such that for every* $n \geq 1$

$$\mathbb{P}_0^\phi \{\hat{\tau} \geq \sigma_{nk}\} \leq p^n, \qquad \phi \in \mathbb{R}_+. \tag{3.19}$$

If $\lambda_1 > \lambda_0$, *then the inequality holds with* $k = 1$ *and* $p = 1 - (\phi_0/\phi_1)^{\lambda_0/(\lambda_1-\lambda_0)}$. *If* $\lambda_1 = \lambda_0$ *and* $\nu_0(\cdot) \not\equiv \nu_1(\cdot)$, *then there exists some* $\delta > 0$ *such that* $q \triangleq \nu_0\left\{y \in \mathbb{R}^d : f(y) \geq 1 + \delta\right\} > 0$, *and (3.19) holds with* $k = \inf\{m \geq 1 :\ (1+\delta)^m \geq \phi_1/\phi_0\}$ *and* $p = 1 - q^k$.

3.15. **Corollary.** *If we let* $n \to \infty$ *in (3.19), then we obtain*

$$\mathbb{P}_0\{\Phi_t \notin (\phi_0, \phi_1) \quad \textit{for some } t \in \mathbb{R}_+\} = 1 \qquad \textit{for every } 0 < \phi_0 < \phi_1 < \infty.$$

4. Solution

We start by describing the *stopping* and *continuation* regions

$$\begin{pmatrix} \mathbf{\Gamma}_n \triangleq \{\phi \in \mathbb{R}_+ : V_n(\phi) = h(\phi)\}, & n \geq 1 \\ \mathbf{\Gamma} \triangleq \{\phi \in \mathbb{R}_+ : V(\phi) = h(\phi)\} & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{C}_n \triangleq \mathbb{R}_+ \setminus \mathbf{\Gamma}_n, & n \geq 1 \\ \mathbf{C} \triangleq \mathbb{R}_+ \setminus \mathbf{\Gamma} & \end{pmatrix}, \tag{4.1}$$

respectively, of the problems in (2.9) and (3.2). By Proposition 3.13 and Corollary 3.9 the optimal stopping time U_0 of the problem in (2.9) and the components $r_n(\cdot) \equiv r_n^0(\cdot)$, $n \geq 1$ of the optimal stopping times S_n^0, $n \geq 1$ of the problems in (3.2) can be rewritten as

$$U_0 = \inf\{t \geq 0 : \Phi_t \in \mathbf{\Gamma}\} \quad \text{and} \quad r_n(\phi) = \inf\{t > 0 : x(t,\phi) \in \mathbf{\Gamma}_{n+1}\},\ \phi \in \mathbb{R}_+,\ n \geq 0. \tag{4.2}$$

We show that each continuation region $\mathbf{C}_n$, $n \geq 1$ and $\mathbf{C}$ is an interval and is contained in the same bounded interval away from the origin. This common structure of the continuation regions guarantees that the convergence of the sequence $\{V_n(\cdot)\}_{n\geq 1}$ to the function $V(\cdot)$ (see Proposition 3.1) is *uniform* on $\mathbb{R}_+$. These results are proved by explicit construction, which later reveals an efficient numerical method to compute the successive approximations $\{V_n(\cdot)\}_{n\geq 1}$ of the value function $V(\cdot)$ in (2.9). The illustration of this method on several examples is deferred to the next section. We conclude this section by describing some ε-optimal strategies to complement the numerical method.

4.1. **Continuation and stopping regions.** Let us show that $V(\cdot) = h(\cdot)$ on $[0, \underline{\xi}] \cup [\overline{\xi}, \infty)$ for some $0 < \underline{\xi} \leq b/a \leq \overline{\xi} < \infty$. Recall from Proposition 3.7 that $V(\cdot)$ satisfies

$$V(\phi) = \inf_{t\geq 0} \left[\int_0^t e^{-\lambda_0 u} [g + \lambda_0 \cdot SV]\big(x(u,\phi)\big) du + e^{-\lambda_0 t} h\big(x(t,\phi)\big) \right], \qquad \phi \in \mathbb{R}_+. \tag{4.3}$$

Since $V(\cdot) \geq -b$ (Propositions 3.4 and 3.7) and $\lambda_1 \geq \lambda_0$, we have $SV(\cdot) \geq -b$ and

$$JV(t,\phi) \geq \varphi(t,\phi) \triangleq \big(1 - e^{-\lambda_0 t}\big) \left(\frac{1}{\lambda_0} - b + \frac{\phi}{\lambda_1} \right) + e^{-\lambda_0 t} h\big(x(t,\phi)\big), \quad t, \phi \in \mathbb{R}_+. \tag{4.4}$$

Denote the exit time of the paths $t \mapsto x(t,\phi)$ of (2.11) from any interval (ψ, ∞) by

$$T(\phi,\psi) \triangleq \inf\{t \geq 0; x(t,\phi) \leq \psi\} = \left[\frac{1}{\lambda_1 - \lambda_0} \cdot \ln\left(\frac{\phi}{\psi}\right) \right]^+, \qquad \phi \in \mathbb{R}_+,\ \psi > 0. \tag{4.5}$$

For every $\phi \geq [\lambda_1 b - (\lambda_1/\lambda_0)] \vee (b/a)$, we have $\inf\{\varphi(t,\phi);\ t \in [0, T(\phi, b/a)]\} = 0 = h(\phi)$, where $\varphi(\cdot,\cdot)$ is the function on the righthand side of (4.4). If $\phi \geq [\lambda_1 b - (\lambda_1/\lambda_0)] \vee (b/a)$ and $t > T(\phi, b/a)$, then $\varphi(t,\phi)$ is greater than or equal to

$$\big(1 - e^{-\lambda_0 T(\phi,b/a)}\big) \left(\frac{1}{\lambda_0} - b + \frac{\phi}{\lambda_1} \right) - e^{-\lambda_0 T(\phi,b/a)} b \geq -b + \big(1 - e^{-\lambda_0 T(\phi,b/a)}\big) \left(\frac{1}{\lambda_0} + \frac{\phi}{\lambda_1} \right).$$

The function of ϕ on the righthand side is increasing and goes to $+\infty$ as $\phi \to +\infty$. If we denote by $\overline{\xi}$ the smallest ϕ such that this function vanishes; i.e.,

$$\overline{\xi} \triangleq \inf \left\{ \phi \geq [\lambda_1 b - (\lambda_1/\lambda_0)] \vee (b/a); \quad \left[1 - \left(\frac{b}{a\phi}\right)^{\lambda_0/(\lambda_1-\lambda_0)} \right] \cdot \left(\frac{1}{\lambda_0} + \frac{\phi}{\lambda_1} \right) \geq b \right\}, \tag{4.6}$$

then $\inf\{\varphi(t,\phi);\ t \in (T(\phi, b/a), \infty)\} \geq 0$ and $V(\phi) = h(\phi)$ for every $\phi \geq \overline{\xi}$.

On the other hand, we have $\varphi(0,\phi) = h(\phi)$ and

$$\frac{\partial \varphi(t,\phi)}{\partial t} = e^{-\lambda_0 t} + \phi e^{-\lambda_1 t}\big(1 - a\lambda_1\big) \geq e^{-\lambda_1 t} \left[1 + \phi\,(1 - a\lambda_1)\right], \quad \forall \phi \in [0, b/a],\ t \in \mathbb{R}_+. \tag{4.7}$$

Thus, the derivative is positive and $V(\phi) \geq \inf_{t\geq 0} \varphi(t,\phi) \geq \varphi(0,\phi) = h(\phi)$ (i.e., $V(\phi) = h(\phi)$) for every $\phi \in [0, \underline{\xi}]$, where

$$\underline{\xi} \triangleq \left(\frac{b}{a}\right) \wedge \left(\frac{1}{(1-a\lambda_1)^-}\right) \qquad (1/0 \equiv +\infty). \tag{4.8}$$

This completes the proof of the first inclusions in (4.9) below. The rest of the inclusions follow from the inequalities $V(\cdot) \leq \ldots \leq V_n(\cdot) \leq V_{n-1}(\cdot) \leq \ldots \leq V_1(\cdot) \leq h(\cdot)$.

4.1. **Proposition.** *Let $0 < \underline{\xi} < b/a < \overline{\xi} < \infty$ be defined as in (4.6) and (4.8). Then*

$$\begin{aligned} &[0, \underline{\xi}] \cup [\overline{\xi}, \infty) \subseteq \mathbf{\Gamma} \subseteq \ldots \subseteq \mathbf{\Gamma}_n \subseteq \mathbf{\Gamma}_{n-1} \subseteq \ldots \subseteq \mathbf{\Gamma}_1, \\ &(\underline{\xi}, \overline{\xi}) \supseteq \mathbf{C} \supseteq \ldots \supseteq \mathbf{C}_n \supseteq \mathbf{C}_{n-1} \supseteq \ldots \supseteq \mathbf{C}_1. \end{aligned} \tag{4.9}$$

4.2. **Corollary.** *Since the functions $V(\cdot)$ and $V_n(\cdot)$, $n \geq 1$ are concave by Propositions 3.4, 3.5, and 3.7, the continuation regions $\mathbf{C}$ and $\mathbf{C}_n$, $n \geq 1$ of (4.1) are bounded open intervals*

$$\mathbf{C} = (\xi_0, \xi_1) \qquad \textit{and} \qquad \mathbf{C}_n = (\xi_0^{(n)}, \xi_1^{(n)}), \quad n \geq 1 \tag{4.10}$$

for some $0 < \underline{\xi} \leq \xi_0 \leq \cdots \leq \xi_0^{(n)} \leq \cdots \leq \xi_0^{(1)} \leq b/a \leq \xi_1^{(1)} \leq \cdots \leq \xi_1^{(n)} \leq \ldots \leq \xi_1 \leq \overline{\xi} < \infty$.

4.3. **Corollary.** (i) We have $V(\cdot) = V_n(\cdot) = h(\cdot)$ for every $n \geq 1$ if and only if either $V_n(b/a) = h(b/a) = 0$ for some $n \geq 1$ or $V(b/a) = h(b/a) = 0$.

(ii) If $\lambda_1 \leq (1/a) + (1/b)$, then $V(\cdot) = V_n(\cdot) = h(\cdot)$ everywhere, and "immediate stopping" is an optimal rule for every problem in (2.9) and (3.2).

(iii) If $(1/a) + (1/b) < \lambda_1 - \lambda_0$, then the continuation regions $\mathbf{C}$ and $\mathbf{C}_n$, $n \geq 1$ of (4.1) are not empty.

Corollary 4.3 is very useful in determining whether the solution is trivial. One can easily calculate $v_1(b/a) = J_0 h(b/a)$ and check if it equals $h(b/a) = 0$ or not.

4.2. **Uniform Convergence.** The optimal stopping time U_0 of Proposition 3.13 for the problem in (2.9) becomes $U_0 = \inf\{t \geq 0 : \Phi_t \notin (\xi_0, \xi_1)\}$ by (4.2) and Corollary 4.2. Therefore, Proposition 3.14 guarantees the existence of some $k \geq 1$ and $p \in (0,1)$ such that $\sup_{\phi \in \mathbb{R}_+} \mathbb{P}_0^\phi \{U_0 \geq \sigma_{nk}\} \leq p^n$ for every $n \geq 1$. Thus, for every $\phi \in \mathbb{R}_+$ and $n \geq 1$

$$\begin{aligned} V_{nk}(\phi) \geq V(\phi) &\geq V_{nk}(\phi) + \mathbb{E}_0^\phi \left[1_{\{U_0 \geq \sigma_{nk}\}} \left(\int_{\sigma_{nk}}^{U_0} g(\Phi_t)dt + h(\Phi_{U_0}) - h(\Phi_{\sigma_{nk}}) \right) \right] \\ &\geq V_{nk}(\phi) + \mathbb{E}_0^\phi \left[1_{\{U_0 \geq \sigma_{nk}\}} h(\Phi_{U_0}) \right] \geq V_{nk}(\phi) - b\mathbb{P}_0^\phi \{U_0 \geq \sigma_{nk}\} \geq V_{nk}(\phi) - bp^n. \end{aligned}$$

Hence, the subsequence $\{V_{nk}(\cdot)\}_{n\in\mathbb{N}}$ converges to $V(\cdot)$ uniformly. Since the sequence $\{V_n(\cdot)\}_{n\in\mathbb{N}}$ is decreasing, it also converges to $V(\cdot)$ uniformly on $\mathbb{R}_+$.

4.4. **Proposition.** *The successive approximations* $\{V_n(\cdot)\}_{n\in\mathbb{N}}$ *in (3.2, 3.8) decrease to the value function* $V(\cdot)$ *of (2.9) uniformly on* $\mathbb{R}_+$. *More precisely, if* $k \geq 1$ *and* $p \in (0,1)$ *are as in Proposition 3.14 when the interval* (ϕ_0, ϕ_1) *is the same as* $\mathbf{C} = (\xi_0, \xi_1)$, *then*

$$bp^n \geq V_{nk}(\phi) - V(\phi) \geq 0, \qquad \phi \in \mathbb{R}_+,\ n \geq 1. \tag{4.11}$$

4.3. **Numerical Solution.** The value function $V(\cdot)$ of (2.9) can be approximated fast and accurately (with a large control on both by (4.11)) by the functions $V_n(\cdot) = v_n(\cdot)$, $n \geq 1$ successively. The successive approximations $\{V_n(\cdot)\}_{n\geq 1}$ of the function $V(\cdot)$ can be calculated numerically by solving the deterministic optimization problems in (3.8). The *smallest* minimizer $r_n(\cdot)$ in (3.11, 4.2) of the deterministic problem $v_{n+1}(\cdot) = J_0 v_n(\cdot) = \inf_{t\in[0,\infty]} Jv_n(t,\cdot)$ can be rewritten by Corollary 4.2 as

$$r_n(\phi) = \inf\{t \geq 0 : \ x(t,\phi) \notin (\xi_0^{(n+1)}, \xi_1^{(n+1)})\}, \qquad n \geq 0. \tag{4.12}$$

The inclusion $\mathbf{C}_{n+1} \subseteq (\underline{\xi}, \overline{\xi})$ of continuation region $\mathbf{C}_{n+1} = (\xi_0^{(n+1)}, \xi_1^{(n+1)})$ by Proposition (4.1) implies that for every $\phi \in \mathbb{R}_+$ the minimizer $r_n(\phi)$ is bounded from above by the exit time $T(\phi, \underline{\xi})$ of $t \mapsto x(t,\phi)$ from the set $(\underline{\xi}, \infty)$; see (4.5). On the other hand, we have $\mathbf{C}_{n+1} \supseteq \mathbf{C}_n = (\xi_0^{(n)}, \xi_1^{(n)})$ and $r_n(\phi) \in \{0\} \cup \big[T\big(\phi, \xi_0^{(n)}\big), T\big(\phi, \underline{\xi}\big)\big]$ for every $\phi \in \mathbb{R}_+$.

The computation of $\{V_n(\cdot)\}_{n\geq 1}$ simplifies if $\lambda_1 = \lambda_0$. In this case, the process Φ of (2.5) is constant between jumps, and $x(t,\phi) = \phi$ for all $t \geq 0$, $\phi \in \mathbb{R}_+$. Therefore, for every bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}$, the function $J_0 w(\phi)$ in (3.4) becomes

$$\inf_{t\in[0,\infty]} \left[\big(1 - e^{-\lambda_0 t}\big) \frac{g(\phi) + \lambda_0 \cdot Sw(\phi)}{\lambda_0} + e^{-\lambda_0 t} h(\phi) \right] = \min\left\{ h(\phi), \frac{g(\phi) + \lambda_0 \cdot Sw(\phi)}{\lambda_0} \right\},$$

and the minimum is attained at $t = 0$ if $h(\phi) \leq (1/\lambda_0)[g + \lambda_0 \cdot Sw](\phi)$ or $t = \infty$ otherwise. The complete numerical method is described in Figure 2.

4.4. **Nearly optimal strategies.** We close this section with the description of two ε-optimal strategies both of which complement the numerical method above.

The first strategy makes use of Propositions 3.5 and 4.4. For any fixed $\varepsilon > 0$, choose $n \geq 1$ by using (4.11) such that $\sup_{\phi\in\mathbb{R}_+} |V(\phi) - V_n(\phi)| \leq \varepsilon/2$. Then the stopping time $S_n^{\varepsilon/2}$ of Proposition 3.5 is ε-optimal:

$$V(\phi) \leq \mathbb{E}_0^\phi \left[\int_0^{S_n^{\varepsilon/2}} g\big(\Phi_t\big) dt + h(\Phi_{\sigma_n^\varepsilon}) \right] \leq V_n(\phi) + \frac{\varepsilon}{2} \leq V(\phi) + \varepsilon, \qquad \phi \in \mathbb{R}_+. \tag{4.13}$$

The stopping rule $S_n^{\varepsilon/2}$ instructs us to wait until the first occurrence of the exit time $r_n^{\varepsilon/2}(\Phi_0)$ in (4.12) and the first jump time σ_1 of the process X. If $r_n^{\varepsilon/2}(\Phi_0)$ occurs first, then we stop.

Step 0: If in (3.4) $J_0 h(b/a) = 0$, then stop: $v_1(\cdot) = v_2(\cdot) = \ldots = h(\cdot)$ by Corollary 4.3(i). Otherwise, determine the interval $(\underline{\xi}, \overline{\xi})$ by using (4.6, 4.8). Initialize $n = 0$, $v_0(\cdot) = h(\cdot)$, $\xi_0^{(0)} = \xi_1^{(0)} = +\infty$, and go to Step 1.

Recall from (3.6) the operator S and from (4.5) the exit time $T(\phi, \psi)$ of $t \mapsto x(t, \phi)$ from the interval (ψ, ∞) for every $\phi, \psi \in \mathbb{R}_+$.

Step 1: For every $\phi \notin (\underline{\xi}, \overline{\xi})$, set $v_{n+1}(\phi)$ to $h(\phi)$. For every $\phi \in (\underline{\xi}, \overline{\xi})$ do the following:

- If $\lambda_1 > \lambda_0$, then set $\mathscr{T}_{n+1}(\phi)$ to $\{0\} \cup [T(\phi, \xi_0^{(n)}), T(\phi, \underline{\xi})]$ and $v_{n+1}(\phi)$ to
$$J_0 v_n(\phi) = \min_{t \in \mathscr{T}_{n+1}(\phi)} \left[\int_0^t e^{-\lambda_0 u} [g + \lambda_0 \cdot S v_n](x(u, \phi)) du + e^{-\lambda_0 t} h(x(t, \phi)) \right].$$
- If $\lambda_1 = \lambda_0$, then set $v_{n+1}(\phi)$ to $\min\{h(\phi), (1/\lambda_0) \cdot [g(\phi) + \lambda_0 \cdot S v_n(\phi)]\}$.

Step 2: Set $\mathbf{C}_{n+1}$ to $\{\phi \in [\underline{\xi}, \overline{\xi}] : v_{n+1}(\phi) = h(\phi)\}$, $\xi_0^{(n+1)}$ and $\xi_1^{(n+1)}$ to the infimum and the supremum of $\mathbf{C}_{n+1}$, respectively. Increase n by one, and go to Step 1.

FIGURE 2. The numerical algorithm to calculate the successive approximations $V_n(\cdot)$, $n \geq 1$ in (3.2, 3.8) of the value function $V(\cdot)$ of (2.9). The infinite loop can be broken according to bounds in (4.11) when n is so large that the desired accuracy is reached.

Otherwise, we continue waiting until the first occurrence of the exit time $r_{n-1}^{\varepsilon/2}(\Phi_{\sigma_1})$ and the next jump at time $\sigma_2 - \sigma_1 = \sigma_1 \circ \theta_{\sigma_1}$. If $r_{n-1}^{\varepsilon/2}(\Phi_{\sigma_1})$ occurs, then we stop. Otherwise, we continue as before. We stop at the nth jump of the process X if we have not stopped yet.

The second ε-optimal stopping rule is easier to implement and is defined by
$$U_{\varepsilon/2}^{(n)} \triangleq \inf\{t \geq 0; \quad h(\Phi_t) \leq V_n(\Phi_t) + \varepsilon/2\}$$
after ε and n are chosen as in the previous paragraph. Since $t \mapsto V(\Phi_t)$ is right-continuous and $|V(\cdot) - V_n(\cdot)| < \varepsilon/2$, we have $V(\Phi_t) \geq h(\Phi_t) - \varepsilon$ at $t = U_{\varepsilon/2}^{(n)}$ on the event $\{U_{\varepsilon/2}^{(n)} < \infty\}$. Then the arguments leading to Proposition 3.12 yields
$$V(\phi) = \mathbb{E}_0^\phi \left[V\big(\Phi_{U_{\varepsilon/2}^{(n)} \wedge \sigma_m}\big) + \int_0^{U_{\varepsilon/2}^{(n)} \wedge \sigma_m} g(\Phi_s) ds \right], \qquad m \geq 1.$$
Because $U_{\varepsilon/2}^{(n)} \leq U_{\varepsilon/2}$ of (3.17), the $\mathbb{P}_0$-expectation of $U_{\varepsilon/2}^{(n)}$ is finite by Proposition 3.13, which implies after obvious modifications that the stopping rule $U_{\varepsilon_2}^{(n)}$ is also ε-optimal:
$$V(\phi) \geq \mathbb{E}_0^\phi \left[\int_0^{U_{\varepsilon_2}^{(n)}} g(\Phi_s) ds + h(\Phi_{U_{\varepsilon_2}^{(n)}}) \right] - \varepsilon.$$

5. Examples

Using the numerical method of Section 4 (see Figure 2) we solve a number of examples with both discrete and continuous mark distributions. The empirical results demonstrate the effect of difference between the alternative hypotheses $(\lambda_0, \nu_0(\cdot))$ and $(\lambda_1, \nu_1(\cdot))$ on the optimal Bayes risk. Finally, we revisit the special case studied by Peskir and Shiryaev (2000).

5.1. **Numerical Examples.** In the first example, the marks $Y_1, Y_2, \ldots$ of the observation process X in (1.1) take values in a space with five elements (labeled with integers 1 through 5 without loss of generality), and the (discrete) mark distributions are

$$\nu_0 = \left\{ \frac{1}{15}, \frac{5}{15}, \frac{4}{15}, \frac{3}{15}, \frac{2}{15} \right\} \qquad \text{and} \qquad \nu_1 = \left\{ \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15} \frac{1}{15} \right\}$$

under the hypotheses H_0 and H_1, respectively. From the plot of ν_0 and ν_1 in the upper left panel of Figure 3, it is easy to see that ν_0 is right-skewed, and ν_1 is left-skewed. We fix $b = 4$, $a = 2$ (the respective costs of Type I and II errors) and $\lambda_0 = 3$ (the arrival rate of X under the null hypothesis H_0). By using the numerical method in Figure 2 on page 15, we solve the sequential hypothesis testing problem in (1.2, 1.4, 2.9) for three different values of the arrival rate λ_1 of X under alternative hypothesis H_1: the panels in the first row of Figure 3 display the successive approximations $\{V_n(\cdot)\}_{n \geq 1}$ for (b) $\lambda_1 = 3$, (c) $\lambda_1 = 6$, (d) $\lambda_1 = 9$. For each case we recalled the (explicit) uniform bound on the difference $|V(\cdot) - V_n(\cdot)|$ in (4.11) and calculated the decreasing sequence $\{V_n(\cdot)\}_{n \geq 1}$ until the maximum difference between two successive approximations is negligible (number of iterations is noted inside each panel). Thus, in each panel the smallest function is the best approximation to the value function $V(\cdot)$ of (2.9). In panel (b), $V_1(\cdot) = V_2(\cdot) = \ldots = V(\cdot) = h(\cdot)$, and "immediate stopping" turns out to be optimal everywhere.

Vertical bars at two horizontal edges of each panel mark the boundaries of each continuation region $\mathbf{C}_n = (\xi_0^{(n)}, \xi_1^{(n)})$, $n \geq 1$ in (4.1, 4.10). The leftmost and the rightmost bars give approximately the boundaries of the continuation region $\mathbf{C} = (\xi_0, \xi_1)$. By Proposition 2.1 an optimal admissible decision rule is to wait until the process Φ of (2.10) leaves the interval (ξ_0, ξ_1) and to choose the null hypothesis H_0 if Φ is less than or equal to b/a upon stopping and the alternative H_1 otherwise. See Section 4.4 for other nearly optimal admissible decision rules and precise error bounds.

If $\lambda_1 = \lambda_0$, then only observed jump sizes will carry useful information to discriminate the hypotheses. If $\lambda_1 > \lambda_0$, then the interarrival times also contain important information. As the difference $\lambda_1 - \lambda_0$ increases, this information becomes more significant, and a lower

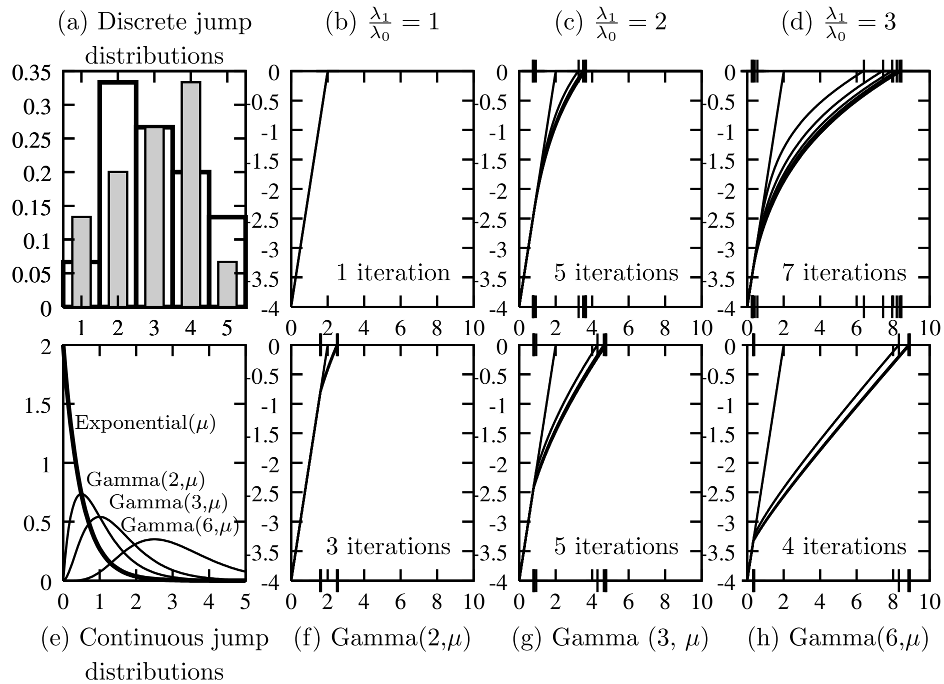

FIGURE 3. Numerical examples solved by successive approximations (see Section 4 and Figure 2). In the upper row, jump distributions ν_0 (in the background) and ν_1 (with filled bars) of (1.2) are discrete as in panel (a). Type I and II error costs ($b = 4$, $a = 2$) and the arrival rate under the null hypothesis H_0 ($\lambda_0 = 3$) are fixed, and the sequential hypothesis problem (1.2, 1.4, 2.9) is solved for different arrival rates λ_1 under the alternative hypothesis H_1. The successive approximations $\{V_n(\cdot)\}$ of the value function $V(\cdot)$ of (2.9) are displayed in the panels of the first row: (b) $\lambda_1 = \lambda_0$, (c) $\lambda_1 = 2\lambda_0$, (d) $\lambda_1 = 3\lambda_0$. The smallest function in each panel is the best approximation of the function $V(\cdot)$ and gets smaller as the difference $\lambda_1 - \lambda_0$ gets larger along (b)-(d). Namely, if the hypotheses are more "separable," then the minimum Bayes risk will be smaller. In (b), $V = h$, and "immediate stopping" is optimal. In the second row, $\lambda_0 = \lambda_1 = 3$, and the distribution ν_0 is exponential with rate $\mu = 2$. The distribution ν_1 under H_1 is Gamma with the same rate μ, but its shape parameter is changed: 2 in (f), 3 in (g), and 6 in (h). The panels display the successive approximations of the value function $V(\cdot)$ and suggest that the smallest Bayes risk decreases as the densities of jump distributions ν_0 and ν_1 are pulled apart more from each other.

optimal Bayes risk is expected. Observe that the approximate value functions in the first row of Figure 3 decrease from (b) to (d). This supports the intuitive remark in light of the relation (2.8) between the value function $V(\cdot)$ and the minimal Bayes risk $U(\cdot)$.

In the next set of examples, the parameters $b = 4$, $a = 2$, $\lambda_0 = \lambda_1 = 3$ are held fixed, and mark distributions are changed. The distributions $\nu_0(\cdot)$ and $\nu_1(\cdot)$ in (1.2) are exponential and Gamma, respectively, with the same rate $\mu = 2$. In the second row of Figure 3, we solve the sequential hypothesis problem when the shape parameter of ν_1 equals 2 in (f), 3 in (g), and 6 in (h). As before each panel displays the decreasing sequence $\{V_n(\cdot)\}$ of successive approximations in (3.8) of the value function $V(\cdot)$ of (2.9) calculated by the numerical method of Figure 2. The lower left panel shows that as the shape parameter of the Gamma distribution increases, the weights assigned to sets by $\nu_0(\cdot)$ and $\nu_1(\cdot)$ become relatively more different. Intuitively, if the distributions under alternative hypotheses differ more from each other, then the jump sizes tend to be more different and carry more information; as a result, the optimal Bayes risk should be smaller. The figures in panels (f) through (h) are consistent with this view: the value functions become more negative as the shape parameters increase.

5.2. **Sequential testing for simple Poisson process.** Peskir and Shiryaev (2000) solved the sequential testing problem of two simple hypotheses about the unknown arrival rate λ of a *simple* Poisson process X; namely, $\nu_0(\cdot) = \nu_1(\cdot) = \delta_{\{1\}}(\cdot)$, and (1.2) becomes

$$H_0 : \lambda = \lambda_0 \qquad \text{and} \qquad H_1 : \lambda = \lambda_1. \tag{5.1}$$

Their method is different from ours. They obtain the optimal admissible decision rule in terms of the posterior probability process $\Pi = \{\Pi_t \triangleq \mathbb{P}(\lambda = \lambda_1 | \mathcal{F}_t), t \geq 0\}$ after solving a suitable free-boundary integro-differential problem similar to (2.14).

The problem (5.1) is a special case of (1.2), and methods of this paper apply. Below we retrieve the main result of Peskir and Shiryaev (2000) and describe in Figure 4 our solution of their numerical example.

Since the jump distribution $\nu(\cdot) = \nu_0(\cdot) = \nu_1(\cdot)$ of the observation process X is known, the Radon-Nikodym derivative $f(\cdot)$ in (2.2) and (2.10) becomes identically one, and the operator in (3.6) simplifies to $Sw(\phi) = w([\lambda_1/\lambda_0]\phi)$, $\phi \in \mathbb{R}_+$ for every bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}$.

5.1. **Proposition.** *Suppose that the observation process X in (1.1) is a simple Poisson process. Then $V(\cdot) = h(\cdot)$ in (2.9), i.e., "immediate stopping" is optimal if and only if*

$$\frac{1}{a} + \frac{1}{b} \geq \lambda_1 - \lambda_0. \tag{5.2}$$

For suitable constants $0 < A^* < B^* < 1$ the stopping time $T^* = \inf\{t \geq 0 : \Pi_t \notin (A^*, B^*)\}$ is optimal for the problem in (2.9), whose continuous value function is continuously differentiable everywhere except at $B^*/(1 - B^*)$.

Since $\Pi_t \triangleq \mathbb{P}\{\lambda = \lambda_1 | \mathcal{F}_t\} = \Phi_t/(1+\Phi_t)$ for every $t \geq 0$, the stopping time $\mathrm{T}^* \triangleq \inf\{t \geq 0 : \Pi_t \notin (A^*, B^*)\} \equiv U_0$ is optimal by Proposition 3.13 and Corollary 4.2 if we set

$$A^* = \frac{\xi_0}{1+\xi_0} \qquad \text{and} \qquad B^* = \frac{\xi_1}{1+\xi_1}. \tag{5.3}$$

Moreover, the value function $V(\cdot)$ is continuous on $\mathbb{R}_+$ by Propositions 3.4 and 3.7 and continuously differentiable everywhere except at ξ_1 by Proposition 6.1 below.

The necessity of the first claim follows from Corollary 4.3 (iii). To prove the sufficiency, it is enough by Corollary 4.3 (i) to show that $V_1(b/a) = 0$. Recall that $f(\cdot) = 1$ and $Sh(\phi) = (a[\lambda_1/\lambda_0]\phi - b)^-$. If we denote by T the first time $T\left(b/a, [\lambda_0/\lambda_1] \cdot [b/a]\right)$ that the

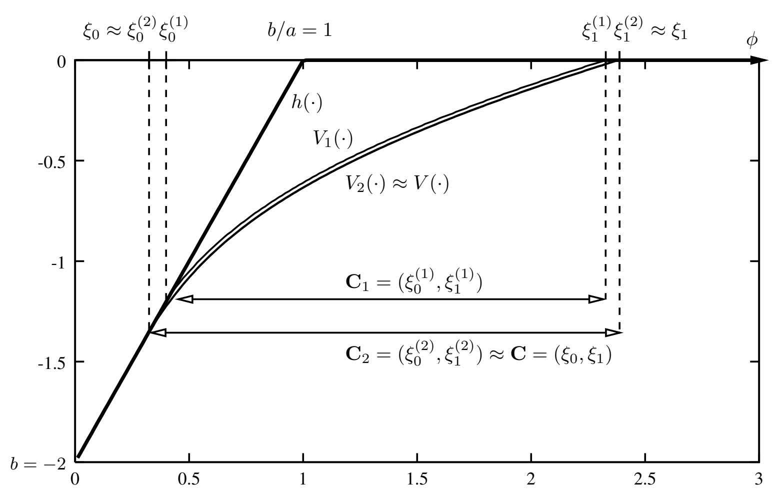

FIGURE 4. Our numerical solution of Peskir and Shiryaev's (2000, Figure 2) sequential testing problem in (5.1) for a *simple* Poisson process ($a = b = 2$, $\lambda_0 = 1$, $\lambda_1 = 5$). This is a special case of (1.2) with *known* jump distribution $\nu(\cdot) = \nu_0(\cdot) = \nu_1(\cdot) = \delta_{\{1\}}(\cdot)$. The method of Figure 2 applies with $f(\cdot) \equiv (d\nu_1/d\nu_0)(\cdot) = 1$ and terminates satisfactorily after only two iterations. The first two successive approximations $V_1(\cdot)$ and $V_2(\cdot)$ in (3.2, 3.8) of the value function $V(\cdot)$ in (2.9) are displayed. The function $V_2(\cdot)$ and the region $\mathbf{C}_2 = (\xi_0^{(2)}, \xi_1^{(2)}) \approx (0.32, 2.39)$ in (4.1, 4.10) are very good approximations of the value function $V(\cdot)$ and the optimal continuation region $\mathbf{C} = (\xi_0, \xi_1)$, respectively. By using (5.3), we calculate the thresholds A^* and B^* of Peskir and Shiryaev's optimal stopping rule T^* in Proposition 5.1. We find $A^* \approx 0.32/(1+0.32) = 0.24$ and $B^* \approx 2.39/(1+2.39) = 0.71$ are very close to $(A^*, B^*) \approx (0.22, 0.70)$ reported by Peskir and Shiryaev.

path $t \mapsto x(t, b/a)$, $t \geq 0$ exits the interval $([\lambda_0/\lambda_1] \cdot [b/a], \infty)$ as in (4.5), then for every $t \geq 0$

$$\frac{d}{dt} Jh(t, b/a) = \left\{ \begin{array}{ll} e^{-\lambda_0 t}[1 + \lambda_0 b] + be^{-\lambda_1 t}\left[(1/a) - \lambda_1\right], & t < T \\ e^{-\lambda_0 t} + (b/a)e^{-\lambda_1 t}, & t \geq T \end{array} \right\} \geq 0,$$

where the inequality from (5.2). Hence, $V_1(b/a) = \inf_{t \geq 0} Jh(t, b/a) = Jh(0, b/a) = 0$.

6. Smoothness and variational inequalities

We start this section with an investigation of the smoothness of the value function $V(\cdot)$ in (2.9). We show that $V(\cdot)$ is not differentiable at the upper boundary ξ_1 of the continuation region $\mathbf{C} = (\xi_0, \xi_1)$. While $V(\cdot)$ is continuously differentiable everywhere else if $\lambda_1 > \lambda_0$, it may not be differentiable at every point of the continuation region $\mathbf{C}$ if $\lambda_1 = \lambda_0$. In the latter case, the lack of differentiability at ξ_1 is transmitted to every point from which the process Φ jumps to ξ_1 with positive probability. Our findings in the case that $\lambda_1 > \lambda_0$ are consistent with the "smooth-fit principle" formulated recently by Alili and Kyprianou (2005): the value function $V(\cdot)$ is continuously differentiable at the lower boundary ξ_0, which is regular for the stopping region, and is not differentiable at the upper boundary ξ_1, which is not regular for the stopping region.

We conclude by showing that the value function $V(\cdot)$ is the unique solution of the variational inequalities in (2.14) in some suitable sense.

6.1. **Smoothness of the value function.** In the following analysis, we will assume that the continuation region $\mathbf{C} = (\xi_0, \xi_1)$ is not empty since the results are immediate from $V(\cdot) = h(\cdot)$ otherwise. By the same token, continuous differentiability of $V(\cdot)$ on the stopping region $\mathbf{\Gamma} = \mathbb{R}_+ \setminus \mathbf{C} \not\ni b/a$ is obvious. Note also that since $V(\cdot)$ is concave, it has left derivative $D^- V(\cdot)$ and right derivative $D^+ V(\cdot)$ everywhere, and they are left- and right-continuous, respectively. Moreover, $D^- V(\cdot) \geq D^+ V(\cdot)$.

Case I: $\lambda_1 > \lambda_0$. To determine the smoothness of $V(\cdot)$ on the continuation region $\mathbf{C}$, we will use the dynamic programming equation given by (3.15, 3.16). Recall from (4.5) the exit time $T(\phi, \psi)$ of the deterministic path $t \mapsto x(t, \phi)$ of (2.11) from the interval (ψ, ∞) for any $\phi \in \mathbb{R}_+, \psi > 0$. Then for any point $\phi \in \mathbf{C} = (\xi_0, \xi_1)$, setting $t = T(\phi, \phi - \delta)$ in (3.16) gives

$$V(\phi) = \int_0^{T(\phi, \phi-\delta)} e^{-\lambda_0 u}\left[g + \lambda_0 SV\right](x(u, \phi))du + e^{-\lambda_0 T(\phi, \phi-\delta)} V(\phi - \delta).$$

We let $\delta \searrow 0$ after subtracting $V(\phi-\delta)$ from both sides and obtain the left-derivative

$$D^- V(\phi) = \frac{1}{(\lambda_1-\lambda_0)\phi} \cdot \left[g(\phi) + \lambda_0 \cdot SV(\phi) - \lambda_0 V(\phi) \right], \quad \phi \in \mathbf{C} = (\xi_0, \xi_1). \tag{6.1}$$

Since $V(\cdot)$ and $SV(\cdot)$ are continuous, the left derivative $D^- V(\cdot)$ is continuous on $\mathbf{C}$ by (6.1). Because $V(\cdot)$ is concave, this implies that $V(\cdot)$ is continuously differentiable on $\mathbf{C}$.

To show that $V(\cdot)$ is differentiable at the lower boundary point ξ_0 of the continuation region $\mathbf{C} = (\xi_0, \xi_1)$, let $\phi \in (\xi_0, \xi_1]$. Then the minimum $V(\phi) = \inf_{t \in [0,\infty]} JV(t,\phi) = JV(r(\phi),\phi)$ is attained at $t = r(\phi) \equiv T(\phi,\xi_0) \in (0,\infty)$ by (3.14, 3.15, 4.10). Since the function $t \mapsto JV(t,\phi)$ is continuously differentiable at $t = T(\phi,\xi_0)$, and $x(T(\phi,\xi_0),\phi) = \xi_0$, we have

$$0 = \frac{\partial}{\partial t} JV(t,\phi) \bigg|_{t=T(\phi,\xi_0)} = e^{-\lambda_0 T(\phi,\xi_0)} \left[(g + \lambda_0 \cdot SV)(\xi_0) - \lambda_0 h(\xi_0) - a\xi_0(\lambda_1-\lambda_0) \right]$$

for every $\phi \in (\xi_0, \xi_1]$. Then $[g+\lambda_0 \cdot SV](\xi_0) - \lambda_0 V(\xi_0) = a\xi_0(\lambda_1-\lambda_0)$ because $V(\xi_0) = h(\xi_0)$, and (6.1) implies

$$D^+ V(\xi_0) = \lim_{\phi \searrow \xi_0} V'(\phi) = \frac{[g+\lambda_0 \cdot SV](\xi_0) - \lambda_0 V(\xi_0)}{\xi_0(\lambda_1-\lambda_0)} = a = D^- V(\xi_0),$$

since $V(\cdot)$ is concave, and $V(\cdot) = h(\cdot)$ on $[0,\xi_0] \not\ni b/a$. Therefore, $V(\cdot)$ is continuously differentiable at ξ_0.

To see that $V(\cdot)$ is not differentiable at ξ_1, first note that $D^+ V(\xi_1) = 0$ since $V(\cdot) = h(\cdot) = 0$ on $[\xi_1,\infty)$, and (6.1) implies

$$D^- V(\xi_1) = \lim_{\phi \nearrow \xi_1} DV(\phi) = \frac{[g+\lambda_0 \cdot SV](\xi_1)}{(\lambda_1-\lambda_0)\xi_1} \tag{6.2}$$

because the left-derivative $D^- V(\cdot)$ of concave function $V(\cdot)$ is left-continuous, and $V(\xi_1) = h(\xi_1) = 0$. By (3.15) and (4.10), we have $r(\xi_1) = T(\xi_1,\xi_0) \in (0,\infty)$, and (3.14) implies

$$\begin{aligned} 0 = V(\xi_1) = JV(T(\xi_1,\xi_0),\xi_1) &= \int_0^{T(\xi_1,\xi_0)} e^{-\lambda_0 u} \left[g + \lambda_0 \cdot SV \right] (x(u,\xi_1)) du + e^{-\lambda_0 T(\xi_1,\xi_0)} h(\xi_0) \\ &\le \frac{[g+\lambda_0 \cdot SV](x(0,\xi_1))}{\lambda_0} + e^{-\lambda_0 T(\xi_1,\xi_0)} h(\xi_0) = \frac{[g+\lambda_0 \cdot SV](\xi_1)}{\lambda_0} + e^{-\lambda_0 T(\xi_1,\xi_0)} h(\xi_0). \end{aligned}$$

The inequality above follows from that the function $[g+\lambda_0 SV](\cdot)$ is increasing and that $u \mapsto x(u,\xi_1)$ is decreasing. It implies that $[g+\lambda_0 SV](\xi_1) > 0$ since $h(\xi_0) < 0$ and $T(\xi_1,\xi_0) < \infty$. Now $D^+ V(\xi_1) > 0 = D^- V(\xi_1)$ by (6.2), and the function $V(\cdot)$ is not differentiable at ξ_1.

Case II: $\lambda_1 = \lambda_0$. In this case, the process Φ of (2.5) remains constant between jumps, and (4.3) reduces to

$$V(\phi) = \min\left\{h(\phi),\ \frac{g(\phi) + \lambda_0 \cdot SV(\phi)}{\lambda_0}\right\}, \qquad \phi \in \mathbb{R}_+. \tag{6.3}$$

Both $\phi = \xi_0$ and $\phi = \xi_1$ satisfy $(1/\lambda_0)g(\phi) + SV(\phi) = h(\phi)$ by the continuity of $V(\cdot)$. Since $SV(\cdot)$ is increasing, we have

$$D^- V(\xi_1) = \lim_{\phi \nearrow \xi_1}\left[\frac{1}{\lambda_0}\frac{g(\xi_1) - g(\phi)}{\xi_1 - \phi} + \frac{SV(\xi_1) - SV(\phi)}{\xi_1 - \phi}\right] \geq \frac{g'(\xi_1)}{\lambda_0} = \frac{1}{\lambda_0} > 0 = D^+ V(\xi_1).$$

Therefore, the function $V(\cdot)$ is not differentiable at ξ_1. On the other hand, for every $\phi \in (\xi_0, \xi_1)$ (6.3) implies

$$\begin{aligned}\frac{V(\phi \pm \delta) - V(\phi)}{\delta} &= \pm\frac{1}{\lambda_0} + \frac{SV(\phi \pm \delta) - SV(\phi)}{\delta}\\ &= \pm\frac{1}{\lambda_0} \pm \int_{\mathbb{R}^d} \nu_0(dy)\, f(y) \cdot \left[\frac{V(z\phi \pm z\delta) - V(z\phi)}{\pm z\delta}\right]\Bigg|_{z=f(y)}.\end{aligned}$$

By Proposition 3.4 the quotient inside the integral is bounded, and $\nu_0\left\{y \in \mathbb{R}^d : f(y) = 0\right\} = \nu_1\left\{y \in \mathbb{R}^d : f(y) = 0\right\} = 0$ since the distributions $\nu_0(\cdot)$ and $\nu_1(\cdot)$ are equivalent. Then the bounded convergence theorem as $\delta \searrow 0$ and the relation $\nu_1(dy) = f(y) \cdot \nu_0(dy)$ imply

$$D^{\pm} V(\phi) = \frac{1}{\lambda_0} + \int_{\mathbb{R}^d} \nu_1(dy)\ D^{\pm} V\left(f(y)\phi\right), \qquad \phi \in \mathbb{R}_+. \tag{6.4}$$

Therefore, the function $[D^- V - D^+ V](\phi)$, $\phi \in \mathbb{R}_+$ is nonnegative and bounded, and satisfies

$$[D^- V - D^+ V](\phi) = \int_{\mathbb{R}^d} \nu_1(dy)\ [D^- V - D^+ V]\left(f(y)\phi\right), \qquad \phi \in \mathbb{R}_+.$$

If $A(\phi) \triangleq \left\{y \in \mathbb{R}^d : f(y)\phi = \xi_1\right\}$, and $\nu_0(A(\phi)) > 0$, then $\nu_1(A(\phi)) = (\xi_1/\phi)\nu_0(A(\phi)) > 0$ and the the last displayed equality imply

$$[D^- V - D^+ V](\phi) \geq \frac{\xi_1}{\phi}\nu_0(A(\phi)) \cdot [D^- V - D^+ V](\xi_1) > 0,$$

Therefore, if $\nu_0(A(\phi)) > 0$, then $V(\cdot)$ is not differentiable at ϕ.

6.1. **Proposition.** *The value function $V(\cdot)$ of (2.9) is not differentiable at the upper bound ξ_1 of the continuation region $\mathbf{C} = (\xi_0, \xi_1)$.*

If $\lambda_1 > \lambda_0$, then the function $V(\cdot)$ is continuously differentiable on $\mathbb{R}_+ \setminus \{\xi_1\}$.

If $\lambda_1 = \lambda_0$ and $\nu_0\{y \in \mathbb{R}^d;\ f(y)\phi = \xi_1\} > 0$, then $V(\cdot)$ is not differentiable at a point $\phi \in \mathbf{C}$. Namely, the lack of differentiability at ξ_1 is transmitted in the continuation region to every point from which the process Φ of (2.5) jumps to ξ_1 with positive probability.

6.2. **Variational inequalities.** We start by showing that the value function $V(\cdot)$ of (2.9) satisfies the variational inequalities in (2.14) at every $\phi \in \mathbb{R}_+$ where $\mathcal{A}V(\phi)$ makes sense.

First assume that $\lambda_1 > \lambda_0$, and that the continuation region $\mathbf{C} = (\xi_0, \xi_1)$ is not empty. Then the derivative $V'(\cdot)$ exists on $\mathbb{R} \setminus \{\xi_1\}$ by Proposition 6.1 and is equal by (6.1) to

$$V'(\phi) = \frac{1}{(\lambda_1 - \lambda_0)\phi} \cdot [g(\phi) + \lambda_0 SV(\phi) - \lambda_0 V(\phi)], \qquad \phi \in \mathbf{C} = (\xi_0, \xi_1), \tag{6.5}$$

which can be rewritten as $\mathcal{A}V(\phi) + g(\phi) = 0$ for every $\phi \in \mathbf{C}$ in terms of the infinitesimal generator $\mathcal{A}H(\phi) = -(\lambda_1 - \lambda_0)\phi H'(\phi) + \lambda_0 SH(\phi) - \lambda_0 H(\phi)$ in (2.13). Therefore, (2.14) is satisfied by $V(\cdot)$ on the region $\mathbf{C}$.

For every $\phi \in \mathbb{R}_+ \setminus (\mathbf{C} \cup \{\xi_1\})$, we have $JV(0, \phi) = h(\phi) = V(\phi) = J_0 V(\phi) = \inf_{t \geq 0} JV(t, \phi)$, and the mapping $t \to JV(t, \phi)$ is continuously differentiable at $t = 0$. Then, the optimality of $t = 0$ implies that for every $\phi \in \mathbb{R}_+ \setminus (\mathbf{C} \cup \{\xi_1\})$

$$0 \leq \left.\frac{dJV(t, \phi)}{dt}\right|_{t=0} = [g + \lambda_0 SV](\phi) - \lambda_0 V(\phi) - (\lambda_1 - \lambda_0)\phi V'(\phi) \equiv \mathcal{A}V(\phi) + g(\phi). \tag{6.6}$$

Since we also have $V(\cdot) \leq h(\cdot)$ on $\mathbb{R}_+$, this implies that $V(\cdot)$ satisfies (2.14) at every $\phi \in \mathbb{R} \setminus \{\xi_1\}$, where $V(\cdot)$ is differentiable.

If the continuation region $\mathbf{C}$ is empty, then $V(\cdot) = h(\cdot)$ on $\mathbb{R}_+$, and (6.6) holds everywhere except at the point $\phi = b/a$, where $h(\cdot)$ is not differentiable. Therefore, $V(\cdot)$ satisfies (2.14) everywhere it is differentiable.

Finally, if $\lambda_1 = \lambda_0$, then the infinitesimal generator in (2.13) becomes $\mathcal{A}H(\phi) = \lambda_0 SH(\phi) - \lambda_0 H(\phi)$. By (6.3) it is immediate that $V(\cdot)$ is a solution of (2.14). The following proposition shows that the value function $V(\cdot)$ of (2.9) is unique solution of (2.14) in a suitable sense.

6.2. **Proposition.** *Let $H : \mathbb{R}_+ \mapsto (-\infty, 0]$ be a continuous and bounded function (which is also continuously differentiable, possibly, except at most finite number of points if $\lambda_1 > \lambda_0$) such that the set $\{\phi \in \mathbb{R}_+ : H(\phi) \neq h(\phi)\}$ is a bounded interval away from the origin. Then $H(\cdot) = V(\cdot)$ on $\mathbb{R}_+$ if at every $\phi \in \mathbb{R}_+$ where $\mathcal{A}H(\cdot)$ is well-defined it satisfies*

$$\min\{\mathcal{A}H(\phi) + g(\phi),\, h(\phi) - H(\phi)\} = 0. \tag{6.7}$$

Proof. For every $\mathbb{F}$-stopping time τ and constant $t \geq 0$, we have

$$\begin{aligned}\mathbb{E}_0^\phi[1_{\{\tau<\infty\}} h(\Phi_{t\wedge\tau})] \geq \mathbb{E}_0^\phi[1_{\{\tau<\infty\}} H(\Phi_{t\wedge\tau})] \geq \mathbb{E}_0^\phi H(\Phi_{t\wedge\tau})& \\ = H(\phi) + \mathbb{E}_0^\phi\left[\int_0^{t\wedge\tau} \mathcal{A}H(\Phi_s)ds\right] \geq H(\phi) - \mathbb{E}_0^\phi\left[\int_0^{t\wedge\tau} g(\Phi_s)ds\right]&.\end{aligned}$$

Above the first and the last inequalities follow from (6.7), the second from $H(\cdot) \leq 0$, and the equality from the chain rule; see Appendix A.3. If the limits of both sides are taken as $t \to \infty$, then the bounded and monotone convergence theorems give

$$\mathbb{E}_0^\phi \left[\int_0^\tau g(\Phi_s) ds + 1_{\{\tau<\infty\}} h(\Phi_\tau) \right] \geq H(\phi), \qquad \phi \in \mathbb{R}_+. \tag{6.8}$$

The infimum over $\mathbb{F}$-stopping times of both sides give the inequality $V(\cdot) \geq H(\cdot)$. We shall prove the equality by showing that the equality holds in (6.8) if τ is the $\mathbb{F}$-stopping time $\tau^* \triangleq \inf\{t \geq 0 : H(\Phi_t) = h(\Phi_t)\}$.

Since the set $\{\phi \in \mathbb{R}_+ : H(\phi) \neq h(\phi)\}$ is a bounded interval away from the origin, the stopping time τ^* is $\mathbb{P}_0$-a.s. finite by Corollary 3.15. For every $t \geq 0$ the chain-rule gives

$$\mathbb{E}_0^\phi[1_{\{\tau^*<\infty\}} H(\Phi_{t\wedge\tau^*})] = H(\phi) + \mathbb{E}_0^\phi \left[\int_0^{t\wedge\tau^*} \mathcal{A}H(\Phi_s) ds \right] = H(\phi) - \mathbb{E}_0^\phi \left[\int_0^{t\wedge\tau^*} g(\Phi_s) ds \right],$$

because $\mathcal{A}H(\Phi_t) + g(\Phi_t) = 0$ on $\{t < \tau^*\}$ by (6.7). Since $H(\cdot)$ is bounded, and $H(\Phi_{\tau^*}) = h(\Phi_{\tau^*})$ on $\{\tau^* < \infty\}$, the bounded and monotone convergence theorems give the desired equality in (6.8) for $\tau = \tau^*$. □

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Appendix

A.1. **Absolutely continuous change of measure.** For $A \in \mathscr{B}(\mathbb{R}_+)$, let $p(\cdot, A) = \{p(t, A)\}_{t\geq 0}$ be a point process defined as

$$p(t, A) \triangleq \sum_{k=1}^\infty 1_{\{\sigma_k \leq t\}} 1_{\{Y_k \in A\}}, \qquad t \geq 0. \tag{A.1}$$

These processes define on $\left(\mathbb{R}_+ \times \mathbb{R}^d, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R}^d)\right)$ the random measure $p((0, t] \times A) \triangleq p(t, A)$, and the process X of (1.1) can be expressed as

$$X_t = X_0 + \int_{(0,t]\times\mathbb{R}^d} y \, p(ds dy), \qquad t \geq 0. \tag{A.2}$$

Under the the probability measure $\mathbb{P}_0$ of Section 2, the process $\{p(t, A);\ t \geq 0\}$ for every fixed $A \in \mathscr{B}(\mathbb{R}^d)$ is a $(\mathbb{P}, \mathbb{F})$-Poisson process with the intensity $\lambda_0 \nu_0(A)$. Equivalently, the process $\{p(t, A) - p_0(t, A);\ t \geq 0\}$ is a $(\mathbb{P}_0, \mathbb{F})$-martingale, where $p_0(t, A) = \lambda_0 t \nu_0(A)$, $t \geq 0$

is the $(\mathbb{P}_0, \mathbb{F})$-compensator of the point process $p(\cdot, A)$ and induces the compensator measure $p_0((0,t] \times A) \triangleq p_0(t, A)$ on $\big(\mathbb{R}_+ \times \mathbb{R}^d, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R}^d)\big)$.

For $y \in \mathbb{R}^d$ and $f(\cdot)$ given in (2.2), we now define $h(y) \triangleq (1-\Theta) + \Theta\, \frac{\lambda_1}{\lambda_0}\, f(y)$. Since Θ is $\mathcal{G}_0$-measurable, the process

$$Z_t \triangleq \exp\left\{ \int_{(0,t]\times\mathbb{R}^d} \ln h(y)\; p(dsdy) - \int_{(0,t]\times\mathbb{R}^d} [h(y)-1]\; p_0(dsdy) \right\}, \qquad t \geq 0$$

is a $(\mathbb{P}_0, \mathbb{G})$-martingale and defines a new probability measure $\mathbb{P}$ on $(\Omega, \vee_{t\geq 0}\mathcal{G}_t)$. Using the definitions of the measures $p(\cdot)$ and $p_0(\cdot)$, it is possible to show that Z_t above is equal to the right-hand side of the expression in (2.3). Then Girsanov theorem for point processes (see, e.g., Jacod and Shiryaev (2003), Brémaud (1981)) implies that the point process $\{p(t,A);\; t \geq 0\}$ has the $(\mathbb{P}, \mathbb{G})$-compensator

$$p_1(t,A) = \int_{(0,t]\times\mathbb{R}^d} h(y)\, p_0(dsdy) = \int_{(0,t]\times\mathbb{R}^d} \left[(1-\Theta)\; \lambda_0 ds\; \nu_0(dy) + \Theta\; \lambda_1 ds\; \nu_1(dy)\right].$$

Therefore, the process $\{p(t,A); t \geq 0\}$ is a $(\mathbb{P}, \mathbb{G})$-Poisson process with the intensity $(1-\Theta)\lambda_0\nu_0(A) + \Theta\lambda_1\nu_1(A)$ for every $A \in \mathscr{B}(\mathbb{R}^d)$, and the process X in (1.1, A.2) is a $(\mathbb{P}, \mathbb{G})$-compound Poisson process with arrival rate $(1-\Theta)\lambda_0 + \Theta\lambda_1$ and mark distribution $(1-\Theta)\nu_0(\cdot) + \Theta\nu_1(\cdot)$.

A.2. **The dynamics of the process Φ in (2.5).** Using (2.4) and (2.5) gives

$$\Phi_t = \Phi_0 \cdot e^{-(\lambda_1-\lambda_0)t} \prod_{k=1}^{N_t} \left[\frac{\lambda_1}{\lambda_0} f(Y_k)\right], \qquad t \geq 0,$$

which implies that Φ is a piecewise-deterministic Markov process. It is also the unique locally bounded solution of the differential equation (see, e.g., Elliott (1982))

$$d\Phi_t = \Phi_{t-}\left[-(\lambda_1-\lambda_0)dt + \int_{\mathbb{R}^d} [f(y)-1] p(dtdy)\right], \tag{A.3}$$

where $p(\cdot)$ is the random measure introduced in Section A.1. The equation (A.3) can be solved pathwise. Let $x(t,\phi)$ be the solution of the ordinary differential equation

$$\frac{d}{dt} x(t,\phi) = -(\lambda_1-\lambda_0) x(t,\phi) \qquad \text{and} \qquad x(0,\phi) = \phi. \tag{A.4}$$

The dynamics in (A.3) imply that between jumps the process Φ follows the integral curves of the differential equation (A.4), and at arrival time σ_n's it is adjusted by the proportion $[\lambda_1/\lambda_0]f(Y_n)$. Since (2.11) is the solution of (A.4), the pathwise solution of (A.3) is (2.10).

A.3. **The infinitesimal generator of the process Φ.** Let $H:\mathbb{R}_+\mapsto\mathbb{R}$ be a bounded function. If $\lambda_1>\lambda_0$, then we also assume that it is continuously differentiable on $\mathbb{R}_+$ except at most finite number of points. Then the dynamics of Φ in (A.3) and the chain rule give (see, e.g., Protter (2004))

$$H(\Phi_t)=H(\Phi_0)-\int_0^t(\lambda_1-\lambda_0)\,\Phi_s H'(\Phi_s)ds+\int_{(0,t]\times\mathbb{R}^d}\left[H\left(\frac{\lambda_1}{\lambda_0}f(y)\Phi_{s-}\right)-H(\Phi_{s-})\right]p(dsdy)$$
$$=H(\Phi_0)+\int_{(0,t]}\Big[-(\lambda_1-\lambda_0)\,\Phi_s H'(\Phi_{s-})+\lambda_0 SH(\Phi_{s-})-\lambda_0 H(\Phi_{s-})\Big]ds+M_t$$

in terms of the integral

$$M_t\triangleq\int_{(0,t]\times\mathbb{R}^d}\left[H\left(\frac{\lambda_1}{\lambda_0}f(y)\Phi_{s-}\right)-H(\Phi_{s-})\right](p(dsdy)-\lambda_0 ds\,\nu_0(dy))$$

with respect to the compensated random measure $p(ds,dy)-\lambda_0 ds\cdot\nu_0(dy)$. Since $H(\cdot)$ is bounded, the process M is a $(\mathbb{P}_0,\mathbb{F})$-martingale, and taking expectations gives $\mathbb{E}_0^{\phi}H(\Phi_t)=H(\phi)+\mathbb{E}_0^{\phi}\int_0^t\mathcal{A}H(\Phi_s)ds$, with the $(\mathbb{P}_0,\mathbb{F})$-infinitesimal generator $\mathcal{A}$ given in (2.13).

If $\lambda_1=\lambda_0$, then the process Φ moves only by jumps. Therefore, differentiability of $H(\cdot)$ is not required, and the infinitesimal generator of (2.13) becomes $\mathcal{A}H(\phi)=\lambda_0 SH(\phi)-\lambda_0 H(\phi)$ in terms of the operator S in (3.6).

A.4. **Proofs of selected results.**

Proof of Proposition 2.1. Let us write $R_{\tau,d}(\pi)$ as $\mathbb{E}\tau+K_{\tau,d}(\pi)$. The independence of Θ and X under $\mathbb{P}_0$ implies

$$\mathbb{E}\tau=\int_0^\infty\mathbb{E}_0\left[Z_t 1_{\{\tau>t\}}\right]dt=\int_0^\infty\mathbb{E}_0\left[1_{\{\tau>t\}}\left(1-\pi+\pi L_t\right)\right]dt=(1-\pi)\mathbb{E}_0^{\frac{\pi}{1-\pi}}\left[\int_0^\tau\left(1+\Phi_t\right)dt\right].$$

Moreover, $K_{\tau,d}(\pi)\triangleq\mathbb{E}\left[\left(a1_{\{d=0,\Theta=1\}}+b1_{\{d=1,\Theta=0\}}\right)1_{\{\tau<\infty\}}\right]$ is the limit as $t\to\infty$ of

$$\mathbb{E}\left[\left(a1_{\{d=0,\Theta=1\}}+b1_{\{d=1,\Theta=0\}}\right)1_{\{\tau\le t\}}\right]=\mathbb{E}_0\left[\left(a\pi L_t 1_{\{d=0\}}+b(1-\pi)1_{\{d=1\}}\right)1_{\{\tau\le t\}}\right].$$

Since $L=\{L_t;\ t\ge 0\}$ is a $(\mathbb{P}_0,\mathbb{F})$-martingale and $\{d=0\}\cap\{\tau\le t\}\in\mathcal{F}_{\tau\wedge t}$, in the last expectation L_t can be replaced with L_τ by optional sampling theorem. By monotone convergence theorem $K_{\tau,d}(\pi)=\mathbb{E}_0\Big[\left(a\pi L_\tau 1_{\{d=0\}}+b(1-\pi)1_{\{d=1\}}\right)1_{\{\tau<\infty\}}\Big]$, which equals

$$b(1-\pi)\mathbb{P}_0\left\{\tau<\infty\right\}+(1-\pi)\,\mathbb{E}_0^{\frac{\pi}{1-\pi}}\left[\left(a\Phi_\tau-b\right)1_{\{d=0,\tau<\infty\}}\right],$$

and (2.6) follows. The inequality $R_{\tau,d}(\pi)\ge R_{\tau,d(\tau)}(\pi)$ follows now from (2.6) and the definition of $d(\tau)$ in (2.7). Finally, (2.8) follows immediately from (2.6). The term $\mathbb{P}_0\{\tau<\infty\}$

multiplying $b(1-\pi)$ is replaced with one without loss of generality since $\mathbb{P}_0\{\tau<\infty\}<1$ implies that $R_{\tau,d}(\pi)=+\infty$ in (2.6) for every admissible d. $\square$

Proof of Proposition 3.1. Since $V(\cdot)\le V_n(\cdot)$ for every $n\ge 1$, we have $V(\cdot)\le \lim_{n\to\infty} V_n(\cdot)$. For the reverse inequality, fix any $\varepsilon>0$. Since $V(\cdot)$ is bounded, there is a stopping time τ_ε such that $\mathbb{E}_0^\phi\left[\int_0^{\tau_\varepsilon} g\left(\Phi_t\right)dt+h\left(\Phi_{\tau_\varepsilon}\right)1_{\{\tau_\varepsilon<\infty\}}\right]\le V(\phi)+\varepsilon$. Because $h(\cdot)\le 0$, $V_n(\phi)\le \mathbb{E}_0^\phi\left[\int_0^{\tau_\varepsilon\wedge\sigma_n} g\left(\Phi_t\right)dt+h\left(\Phi_{\tau_\varepsilon\wedge\sigma_n}\right)\right]\le \mathbb{E}_0^\phi\left[\int_0^{\tau_\varepsilon\wedge\sigma_n} g\left(\Phi_t\right)dt+h\left(\Phi_{\tau_\varepsilon\wedge\sigma_n}\right)1_{\{\tau_\varepsilon<\infty\}}\right]$ for every $n\ge 1$. As n tends to ∞, we have $\sigma_n\to\infty$ and $\Phi_{\tau_\varepsilon\wedge\sigma_n}\to\Phi_{\tau_\varepsilon}$ on $\{\tau_\varepsilon<\infty\}$ $\mathbb{P}_0$-almost surely. Therefore, the monotone and bounded convergence theorems imply that $\lim_{n\to\infty}V_n(\phi)\le \mathbb{E}_0^\phi\left[\int_0^{\tau_\varepsilon} g\left(\Phi_t\right)dt+h\left(\Phi_{\tau_\varepsilon}\right)1_{\{\tau_\varepsilon<\infty\}}\right]\le V(\phi)+\varepsilon$. Since ε is arbitrary, the result follows. $\square$

Proof of Proposition 3.4. Note that $v_1(\phi)=J_0h(\phi)\le Jh(0,\phi)=h(\phi)=v_0(\phi)$. If $v_n(\cdot)\le v_{n-1}(\cdot)$ for some $n\ge 0$, then $v_{n+1}(\phi)=J_0v_n(\phi)\le J_0v_{n-1}(\phi)=v_n(\phi)$. Therefore, the sequence $\{v_n(\cdot)\}_{n\ge 0}$ is decreasing. Since $v_0\equiv h$ is concave, every $v_n(\cdot)$, $n\ge 0$ is concave by (3.8) and Remark 3.3.

The inequalities $-b\le v(\cdot)\le v_n(\cdot)\le h(\cdot)\le 0$ follow from Remark 3.3, that $v_0(\cdot)\equiv h(\cdot)\ge -b$ is bounded, and (3.8). For $n=0$, we have $v_0(0)=h(0)=-b$. Suppose that $v_n(0)=-b$ for some $n\ge 0$. Since $Sv_n(0)=v_n(0)$ and $x(t,0)=0$ for every $t\ge 0$, we have $v_{n+1}(0)=\inf_{t\in[0,\infty]}Jv_n(t,0)$, which equals

$$\inf_{t\ge 0}\left(\left[1+\lambda_0 v_n(0)\right]\frac{1-e^{-\lambda_0 t}}{\lambda_0}-e^{-\lambda_0 t}b\right)=\frac{1-\lambda_0 b}{\lambda_0}+\inf_{t\ge 0}\left[e^{-\lambda_0 t}\left(-b-\frac{1-\lambda_0 b}{\lambda_0}\right)\right]=-b.$$

Hence, $v_n(0)=-b$ for every $n\ge 0$ by induction, and $v(0)=\lim_{n\to\infty}v_n(0)=-b$.

The remainder follow easily from the concavity of $v_n(\cdot)$, $n\ge 1$ (see, e.g., Protter and Morrey (1991)) and that they can be extended on the set $\{\phi\in\mathbb{R}_+:\phi\ge -1\}\supset\mathbb{R}_+$. $\square$

Proof of Proposition 3.5. First, we shall establish the inequality

$$\mathbb{E}_0^\phi\int_0^{\tau\wedge\sigma_n} g\big(\Phi_t\big)dt+h\big(\Phi_{\tau\wedge\sigma_n}\big)\ge v_n(\phi),\qquad \tau\in\mathbb{F},\ \phi\in\mathbb{R}_+ \tag{A.5}$$

for every $n\ge 0$, by proving inductively on $k=1,\dots,n+1$ that

$$\begin{aligned}\text{(A.6)}\quad &\mathbb{E}_0^\phi\left[\int_0^{\tau\wedge\sigma_n} g\big(\Phi_t\big)dt+h\big(\Phi_{\tau\wedge\sigma_n}\big)\right]\\ &\ge \mathbb{E}_0^\phi\left[\int_0^{\tau\wedge\sigma_{n-k+1}} g\big(\Phi_t\big)dt+1_{\{\tau<\sigma_{n-k+1}\}}h\big(\Phi_\tau\big)+1_{\{\tau\ge\sigma_{n-k+1}\}}v_{k-1}\big(\Phi_{\sigma_{n-k+1}}\big)\right]=:RHS_{k-1}.\end{aligned}$$

Observe that (A.5) follows from (A.6) when we set $k=n+1$.

If $k=1$, then the inequality (A.6) is satisfied as an equality since $v_0 \equiv h$. Suppose that (A.6) holds for some $1 \le k < n+1$. We shall prove that it must also hold when k is replaced with $k+1$. Let us denote the righthand side of (A.6) by RHS_{k-1}, and rewrite it as

$$\text{(A.7)} \quad RHS_{k-1} = RHS_{k-1}^{(1)} + RHS_{k-1}^{(2)} \triangleq \mathbb{E}_0^\phi\left[\int_0^{\tau\wedge\sigma_{n-k}} g(\Phi_t)dt + 1_{\{\tau<\sigma_{n-k}\}}h(\Phi_\tau)\right]$$
$$+ \mathbb{E}_0^\phi\left[1_{\{\tau\ge\sigma_{n-k}\}}\left(\int_{\sigma_{n-k}}^{\tau\wedge\sigma_{n-k+1}} g(\Phi_t)dt + 1_{\{\tau<\sigma_{n-k+1}\}}h(\Phi_\tau) + 1_{\{\tau\ge\sigma_{n-k+1}\}}v_{k-1}(\Phi_{\sigma_{n-k+1}})\right)\right]$$

By Lemma 3.6, there is an $\mathcal{F}_{\sigma_{n-k}}$-measurable random variable R_{n-k} such that $\tau\wedge\sigma_{n-k+1} = (\sigma_{n-k}+R_{n-k})\wedge\sigma_{n-k+1}$ $\mathbb{P}_0$-almost surely on $\{\tau \ge \sigma_{n-k}\}$. By the strong Markov property $\mathbb{E}_0^\phi\left\{1_{\{\tau\ge\sigma_{n-k}\}}Jv_{k-1}(R_{n-k},\Phi_{\sigma_{n-k}})\right\} \ge \mathbb{E}_0^\phi\left[1_{\{\tau\ge\sigma_{n-k}\}}v_k(\Phi_{\sigma_{n-k}})\right]$. From (A.6) and (A.7)

$$\mathbb{E}_0^\phi\left[\int_0^{\tau\wedge\sigma_n} g(\Phi_t)dt + h(\Phi_{\tau\wedge\sigma_n})\right] \ge RHS_{k-1} = \mathbb{E}_0^\phi\left[\int_0^{\tau\wedge\sigma_{n-k}} g(\Phi_t)dt + 1_{\{\tau<\sigma_{n-k}\}}h(\Phi_\tau)\right]$$
$$+ RHS_{k-1}^{(2)} \ge \mathbb{E}_0^\phi\left[\int_0^{\tau\wedge\sigma_{n-k}} g(\Phi_t)dt + 1_{\{\tau<\sigma_{n-k}\}}h(\Phi_\tau) + 1_{\{\tau\ge\sigma_{n-k}\}}v_k(\Phi_{\sigma_{n-k}})\right] = RHS_k.$$

This completes the proof of (A.6) by induction on k, and (A.5) follows by setting $k=n+1$ in (A.6). When we take the infimum of both sides in (A.5), we obtain $V_n(\cdot) \ge v_n(\cdot)$, $n \ge 1$.

The opposite inequality $V_n(\cdot) \le v_n(\cdot)$, $n\ge 1$ follows from (3.9) since every $\mathbb{F}$-stopping time S_n^ε is less than or equal to σ_n, $\mathbb{P}_0$-a.s by construction. Therefore, we only need to establish (3.9). We will prove it by induction on $n \ge 1$. For $n=1$, the lefthand side of (3.9) becomes

$$\mathbb{E}_0^\phi\left[\int_0^{S_1^\varepsilon} g(\Phi_t)dt + h(\Phi_{S_1^\varepsilon})\right] = \mathbb{E}_0^\phi\left[\int_0^{r_0^\varepsilon(\phi)\wedge\sigma_1} g(\Phi_t)dt + h(\Phi_{r_0^\varepsilon(\phi)\wedge\sigma_1})\right] = Jv_0(r_0^\varepsilon(\phi),\phi).$$

Since $Jv_0(r_0^\varepsilon(\phi),\phi) \le J_0v_0(\phi)+\varepsilon$ by the definition of $r_0^\varepsilon(\cdot,\cdot)$ and Remark 3.2, (3.9) holds for $n=1$.

Suppose that (3.9) holds for every $\varepsilon>0$ for some $n\in\mathbb{N}$. We will prove that it also holds when n is replaced with $n+1$. Since $S_{n+1}^\varepsilon\wedge\sigma_1 = r_n^{\varepsilon/2}(\Phi_0)\wedge\sigma_1$, $\mathbb{P}_0$-a.s., by the strong Markov property $\mathbb{E}_0^\phi\left[\int_0^{S_{n+1}^\varepsilon} g(\Phi_t)dt + h(\Phi_{S_{n+1}^\varepsilon})\right]$ equals

$$\mathbb{E}_0^\phi\left[\int_0^{r_n^{\varepsilon/2}(\phi)\wedge\sigma_1} g(\Phi_t)dt + 1_{\{r_n^{\varepsilon/2}(\phi)<\sigma_1\}}h(\Phi_{r_n^{\varepsilon/2}(\phi)})\right] + \mathbb{E}_0^\phi\left[1_{\{r_n^{\varepsilon/2}(\phi)\ge\sigma_1\}}f_n(\Phi_{\sigma_1})\right]$$

where $f_n(\phi) \triangleq \mathbb{E}_0^\phi[\int_0^{S_n^{\varepsilon/2}} g(\Phi_t)dt + h(\Phi_{S_n^{\varepsilon/2}})] \le v_n(\phi)+\frac{\varepsilon}{2}$ by the induction hypothesis. Therefore, it is less than or equal to $Jv_n(r_n^{\varepsilon/2}(\phi),\phi) \le v_{n+1}(\phi)+\varepsilon/2$ by the definition of $r_n^{\varepsilon/2}$ and Remark 3.2. Therefore, (3.9) holds when n is replaced with $n+1$. □

Proof of Proposition 3.7. The first claim follows immediately from Propositions 3.1 and 3.5. For the second claim, since the sequence $\{v_n(\cdot)\}_{n\geq 0}$ is decreasing and bounded by Proposition 3.4, the dominated convergence theorem implies

$$V(\phi) = v(\phi) = \inf_{n\geq 1} v_n(\phi) = \inf_{n\geq 1} J_0 v_{n-1}(\phi) = \inf_{n\geq 1}\inf_{t\geq 0} J v_{n-1}(t,\phi) = \inf_{t\geq 0}\inf_{n\geq 1} J v_{n-1}(t,\phi)$$
$$= \inf_{t\geq 0}\left[\int_0^t e^{-\lambda_0 u}[g+\lambda_0\cdot Sv]\big(x(u,\phi)\big)du + e^{-\lambda_0 t}h\big(x(t,\phi)\big)\right] = J_0 v(\phi) = J_0 V(\phi).$$

Let $U:\mathbb{R}_+\to\mathbb{R}$ be a solution of $U = J_0 U$ smaller than or equal to h. Since $U\leq h$, we have $U = J_0 U \leq J_0 h = v_1$ by Remark 3.3. Assume $U\leq v_n$ for some $n\geq 0$, then similarly $U = J_0 U\leq J_0 v_n = v_{n+1}$. By induction we have $U\leq v_n$, for every $n\geq 1$, implying $U\leq \lim_{n\to\infty} v_n = V$. □

Proof of Lemma 3.8. Let us fix a constant $u\geq t$ and $\phi\in\mathbb{R}_+$. Then

$$\text{(A.8)}\quad Jw(u,\phi) = \mathbb{E}_0^\phi\left[\int_0^{u\wedge\sigma_1} g(\Phi_s)ds + 1_{\{u<\sigma_1\}}h(\Phi_u) + 1_{\{u\geq\sigma_1\}}w(\Phi_{\sigma_1})\right]$$
$$= \mathbb{E}_0^\phi\left[\int_0^{t\wedge\sigma_1} g(\Phi_s)ds + 1_{\{u<\sigma_1\}}h(\Phi_u) + 1_{\{u\geq\sigma_1\}}w(\Phi_{\sigma_1})\right] + \mathbb{E}^\phi\left[1_{\{\sigma_1>t\}}\int_t^{u\wedge\sigma_1} g(\Phi_s)ds\right].$$

On the event $\{\sigma_1>t\}$, we have $u\wedge\sigma_1 = t + [(u-t)\wedge\sigma_1\circ\theta_t]$. Therefore, the strong Markov property of Φ applied to the second integral above gives

$$\text{(A.9)}\quad \mathbb{E}_0^\phi\left[1_{\{\sigma_1>t\}}\int_t^{u\wedge\sigma_1} g(\Phi_s)ds\right] = \mathbb{E}_0^\phi\left[1_{\{\sigma_1>t\}}\mathbb{E}_0^{\Phi_t}\left[\int_0^{(u-t)\wedge\sigma_1} g(\Phi_s)ds\right]\right]$$
$$= \mathbb{E}_0^\phi\left[1_{\{\sigma_1>t\}}\Big(Jw(u-t,\Phi_t) - \mathbb{E}_0^{\Phi_t}\left[1_{\{u-t<\sigma_1\}}h(\Phi_{u-t}) + 1_{\{u-t\geq\sigma_1\}}w(\Phi_{\sigma_1})\right]\Big)\right]$$
$$= e^{-\lambda_0 t}\, Jw\big(u-t,x(t,\phi)\big) - \mathbb{E}_0^\phi\left[1_{\{u<\sigma_1\}}h(\Phi_u)\right] - \mathbb{E}_0^\phi\left[1_{\{\sigma_1>t\}}1_{\{u\geq\sigma_1\}}w(\Phi_{\sigma_1})\right].$$

The second equality follows from the definition of Jw in (3.3); the third from (2.10) and the strong Markov property. Simplifications after substituting (A.9) into (A.8) give $Jw(u,\phi) = Jw(t,\phi) + e^{-\lambda_0 t}\left[Jw\big(u-t,x(t,\phi)\big) - h\big(x(t,\phi)\big)\right]$. Finally, taking the infimum of both sides over $u\in[t,+\infty]$ gives (3.10). □

Proof of Corollary 3.9. Let us fix $\phi\in\mathbb{R}_+$, and denote $r_n(\phi)$ by r_n. By Remark 3.2, we have $Jv_n(r_n,\phi) = J_0 v_n(\phi) = J_{r_n} v_n(\phi)$.

- Suppose first $r_n<\infty$. Since $J_0 v_n = v_{n+1}$, taking $t = r_n$ and $w = v_n$ in Lemma 3.8 gives

$$Jv_n(r_n,\phi) = J_{r_n}v_n(\phi) = Jv_n(r_n,\phi) + e^{-\lambda_0 r_n}\Big[v_{n+1}(x(r_n,\phi)) - h\big(x(r_n,\phi)\big)\Big].$$

Therefore, $v_{n+1}(x(r_n,\phi)) = h\big(x(r_n,\phi)\big)$.

If $0 < t < r_n$, then $Jv_n(t,\phi) > J_0 v_n(\phi) = J_{r_n} v_n(\phi) = J_t v_n(\phi)$ since $u \mapsto J_u v_n(\phi)$ is nondecreasing. Taking $t \in (0, r_n)$ and $w = v_n$ in Lemma 3.8 imply

$$J_0 v_n(\phi) = J_t v_n(\phi) = Jv_n(t,\phi) + e^{-\lambda_0 t}\Big[v_{n+1}(x(t,\phi)) - h\big(x(t,\phi)\big)\Big].$$

Therefore, $v_{n+1}(x(t,\phi)) < h\big(x(t,\phi)\big)$ for every $t \in (0, r_n)$, and (3.11) follows.

• Suppose now that $r_n = \infty$. Then we have $v_{n+1}(x(t,\phi)) < h\big(x(t,\phi)\big)$ for every $t \in (0,\infty)$ by the same argument in the last paragraph above. Hence, $\{t > 0 : v_{n+1}(x(t,\phi)) = h\big(x(t,\phi)\big)\} = \varnothing$, and (3.11) still holds. □

Proof of Proposition 3.12. First, let us show (3.18) for $n = 1$. Fix $\varepsilon \ge 0$ and $\phi \in \mathbb{R}_+$. By Lemma 3.6, there exists a constant $u \in [0,\infty]$ such that $U_\varepsilon \wedge \sigma_1 = u \wedge \sigma_1$. Then

$$\mathbb{E}_0^\phi[M_{U_\varepsilon \wedge \sigma_1}] = JV(u,\phi) + e^{-\lambda_0 u}\big[V\big(x(u,\phi)\big) - h\big(x(u,\phi)\big)\big] = J_u V(\phi), \tag{A.10}$$

where the first equality follows from (3.3) and (2.10), and the second from (3.13).

Fix any $t \in [0,u)$. By (3.13) and (2.10),

$$JV(t,\phi) = J_t V(\phi) - e^{-\lambda_0 t}\big[V\left(x(t,\phi)\right) - h\big(x(t,\phi)\big)\big] \ge J_0 V(\phi) - \mathbb{E}_0^\phi\left[1_{\{\sigma_1 > t\}}\left(V(\Phi_t) - h(\Phi_t)\right)\right].$$

On the event $\{\sigma_1 > t\}$, we have $U_\varepsilon > t$ (otherwise, $U_\varepsilon \le t < \sigma_1$ would imply $U_\varepsilon = u \le t$, which contradicts with initial choice of $t < u$). Thus, $V(\Phi_t) < h(\Phi_t) - \varepsilon$ on $\{\sigma_1 > t\}$, and $JV(t,\phi) > J_0 V(\phi) + \varepsilon\, e^{-\lambda_0 t} \ge J_0 V(\phi)$ for every $t \in [0,u)$. Therefore, $J_0 V(\phi) = J_u V(\phi)$, and (A.10) implies $\mathbb{E}_0^\phi[M_{U_\varepsilon \wedge \sigma_1}] = J_u V(\phi) = J_0 V(\phi) = V(\phi) = \mathbb{E}_0^\phi[M_0]$. This completes the proof of (3.18) for $n = 1$.

Now suppose that (3.18) holds for some $n \ge 1$, and let us show the same equality for $n+1$. Note that $\mathbb{E}_0^\phi[M_{U_\varepsilon \wedge \sigma_{n+1}}] = \mathbb{E}_0^\phi[1_{\{U_\varepsilon < \sigma_1\}} M_{U_\varepsilon}] + \mathbb{E}_0^\phi[1_{\{U_\varepsilon \ge \sigma_1\}} M_{U_\varepsilon \wedge \sigma_{n+1}}]$ equals

$$\mathbb{E}_0^\phi\left[1_{\{U_\varepsilon < \sigma_1\}} M_{U_\varepsilon} + 1_{\{U_\varepsilon \ge \sigma_1\}} \int_0^{\sigma_1} g(\Phi_s) ds\right] + \mathbb{E}_0^\phi\left[1_{\{U_\varepsilon \ge \sigma_1\}} \left\{\int_{\sigma_1}^t g(\Phi_s) ds + V(\Phi_t)\right\}\Big|_{t = U_\varepsilon \wedge \sigma_{n+1}}\right].$$

Since $U_\varepsilon \wedge \sigma_{n+1} = \sigma_1 + [(U_\varepsilon \wedge \sigma_n) \circ \theta_{\sigma_1}]$ on the event $\{U_\varepsilon \ge \sigma_1\}$, the strong Markov property of Φ at the stopping time σ_1 and the induction hypothesis imply that the last expectation equals $\mathbb{E}_0^{\Phi_{\sigma_1}}\left[\int_0^{U_\varepsilon \wedge \sigma_n} g(\Phi_s) ds + V(\Phi_{U_\varepsilon \wedge \sigma_n})\right] = V(\Phi_{\sigma_1})$, and $\mathbb{E}_0^\phi[M_{U_\varepsilon \wedge \sigma_{n+1}}] = \mathbb{E}_0^\phi[1_{\{U_\varepsilon < \sigma_1\}} M_{U_\varepsilon}] + \mathbb{E}_0^\phi[1_{\{U_\varepsilon \ge \sigma_1\}} M_{\sigma_1}] = \mathbb{E}^\phi[M_{U_\varepsilon \wedge \sigma_1}] = \mathbb{E}^\phi[M_0]$, where the last equality was proved above. This concludes the proof of the induction step. □

Proof of Proposition 3.13. Recall from Propositions 3.4 and 3.7 that $-b \le V(\cdot) \le h(\cdot) \le 0$. Since $g(\cdot) \ge 1$, using (3.18) we have $0 \ge V(\phi) = \mathbb{E}_0^\phi\left[V(\Phi_{U_\varepsilon \wedge \sigma_n}) + \int_0^{U_\varepsilon \wedge \sigma_n} g(\Phi_s) ds\right] \ge \mathbb{E}_0^\phi\left[-b + U_\varepsilon \wedge \sigma_n\right]$. So $b \ge \mathbb{E}_0^\phi\left[U_\varepsilon \wedge \sigma_n\right]$ for every $n \ge 1$, and U_ε has finite expectation by the monotone convergence theorem. For the second claim, note that the sequence of random variables $\int_0^{U_\varepsilon\wedge\sigma_n} g(\Phi_s)ds + V(\Phi_{U_\varepsilon\wedge\sigma_n}) \geq -b$, $n\geq 1$ is bounded from below. Since U_ε is finite $\mathbb{P}_0$-a.s., the remark before Proposition 3.13, (3.18) and Fatou's Lemma give the inequality $V(\phi)\geq \mathbb{E}_0^\phi\left[\int_0^{U_\varepsilon} g(\Phi_s)ds + 1_{\{U_\varepsilon<\infty\}}V(\Phi_{U_\varepsilon})\right]$, which is equal to

$$\begin{aligned}
&\mathbb{E}_0^\phi\left[\int_0^{U_\varepsilon} g(\Phi_s)ds + 1_{\{U_\varepsilon<\infty\}}h(\Phi_{U_\varepsilon})\right] + \mathbb{E}_0^\phi\left[1_{\{U_\varepsilon<\infty\}}\big(V(\Phi_{U_\varepsilon})-h(\Phi_{U_\varepsilon})\big)\right]\\
&\geq \mathbb{E}_0^\phi\left[\int_0^{U_\varepsilon} g(\Phi_s)ds + 1_{\{U_\varepsilon<\infty\}}h(\Phi_{U_\varepsilon})\right] - \varepsilon \qquad \text{for every} \quad \phi\in\mathbb{R}_+. \qquad \square
\end{aligned}$$

Proof of Proposition 3.14. It is enough to prove only for $n=1$ that (3.19) holds for some $k\geq 1$ and $p\in(0,1)$. Indeed, if (3.19) holds for some $n\geq 1$, then the strong Markov property will imply that $\mathbb{P}_0^\phi\left\{\hat\tau\geq\sigma_{(n+1)k}\right\} = \mathbb{E}_0^\phi\left[1_{\{\hat\tau\geq\sigma_{nk}\}}\mathbb{P}_0^{\Phi_{\sigma_{nk}}}\left\{\hat\tau\geq\sigma_k\right\}\right] \leq \mathbb{P}_0^\phi\left\{\hat\tau\geq\sigma_{nk}\right\}\cdot p\leq p^{n+1}$.

If $\Phi_0=\phi\notin(\phi_0,\phi_1)$, then clearly $\mathbb{P}_0^\phi\{\hat\tau\geq\sigma_m\}=0$ for every $m\geq 1$, and the inequality (3.19) holds for $n=1$ and for any $k\geq 1$ and $p\in(0,1)$. Suppose that $\Phi_0=\phi\in(\phi_0,\phi_1)$, and $\lambda_1>\lambda_0$. In terms of the exit time $T(\phi,\phi_0)\triangleq\inf\{t\geq 0;\ x(t,\phi)\leq\phi_0\} = -\ln(\phi_0/\phi)/(\lambda_1-\lambda_0)$ of the deterministic path $t\mapsto x(t,\phi)$ in (2.11), $\mathbb{P}_0^\phi\{\hat\tau\geq\sigma_1\}=1-\mathbb{P}_0^\phi\{\hat\tau<\sigma_1\}$ equals

$$1-\mathbb{P}_0^\phi\{T(\phi,\phi_0)<\sigma_1\} = 1-e^{-\lambda_0 T(\phi,\phi_0)} = 1-\left(\frac{\phi_0}{\phi}\right)^{\lambda_0/(\lambda_1-\lambda_0)} \leq 1-\left(\frac{\phi_0}{\phi_1}\right)^{\lambda_0/(\lambda_1-\lambda_0)},$$

since the first arrival time σ_1 of X has exponential distribution with rate λ_0 under $\mathbb{P}_0$. Hence, if $\lambda_1>\lambda_0$, then (3.19) holds for $n=1$ with $k=1$ and $p=1-(\phi_0/\phi_1)^{\lambda_0/(\lambda_1-\lambda_0)}\in(0,1)$.

If $\lambda_1=\lambda_0$, then Φ stays constant between jumps and $T(\phi,\phi_0)=\infty$, and a new argument is needed. If the distributions $\nu_0(\cdot)$ and $\nu_1(\cdot)$ are not identical, then $\nu_0\{y\in\mathbb{R}^d: f(y)\neq 1\}>0$ in (2.2), and the identity $\int_{\mathbb{R}^d} f(y)\nu_0(dy)=1$ implies that there is some $\delta>0$ such that $\mathbb{P}_0\{f(Y)\geq 1+\delta\} = \nu_0\left\{y\in\mathbb{R}^d: f(y)\geq 1+\delta\right\}>0$. Fix such δ and define $k\triangleq\inf\{m\geq 1:\ (1+\delta)^m\geq\phi_1/\phi_0\}<\infty$. Finally, the dynamics of Φ in (2.10) imply that $\{\hat\tau\geq\sigma_k\}\subseteq\Omega\setminus\left(\bigcap_{i=1}^k\{f(Y_i)\geq 1+\delta\}\right)$ for every $\Phi_0=\phi\in(\phi_0,\phi_1)$, and $\mathbb{P}_0^\phi\{\hat\tau\geq\sigma_k\}\leq p\triangleq 1-(\mathbb{P}_0\{f(Y)\geq 1+\delta\})^k\in(0,1)$. This completes the proof of (3.19) for $n=1$ if $\lambda_1=\lambda_0$. $\square$

Proof of Corollary 4.3. (i) Necessity is obvious. Since $V(\cdot)\leq V_n(\cdot)\leq V_1(\cdot)\leq h(\cdot)$, it is enough to prove the sufficiency when $V_1(b/a)=h(b/a)$. This implies $V_1(\cdot)=h(\cdot)$ on $[0,\underline{\xi}]\cup\{b/a\}\cup[\overline{\xi},\infty)$ by Proposition 4.1. But the latter and the concavity imply that $V_1(\cdot)=h(\cdot)$ everywhere. If $V_n(\cdot)=h(\cdot)$ for some $n\geq 1$, then $V_{n+1}(\cdot)=J_0V_n(\cdot)=J_0h(\cdot)=V_1(\cdot)=h(\cdot)$. By induction $V_n(\cdot)=h(\cdot)$ for every $n\geq 1$, and $V(\cdot)=\lim V_n(\cdot)=h(\cdot)$.

(ii) If $\lambda_1 \le (1/a) + (1/b)$, then the righthand side of (4.7) equals zero when $\phi = b/a$, and $t \mapsto \varphi(t, b/a)$ is nondecreasing. Therefore, $V(b/a) \ge \inf_{t\ge 0} \varphi(t, b/a) = \varphi(0, b/a) = h(b/a)$ by (4.4). Thus $V(b/a) = h(b/a)$, and the conclusion follows from the first part.

(iii) Since $h(\cdot)$ is concave, we have $Sh(\phi) \le H(\phi) \triangleq h((\lambda_1/\lambda_0)\phi)$ by Jensen's inequality. If $T \triangleq T(b/a, [\lambda_0/\lambda_1] \cdot [b/a])$ as in (4.5), then $V_1(b/a) = v_1(b/a) = \inf_{t\in[0,\infty]} Jh(t, b/a) \le$

$$\inf_{t\le T} \left[\int_0^t e^{-\lambda_0 u} \left[g + \lambda_0 \cdot H \right] (x(u, b/a)) du + e^{-\lambda_0 t} h\left(x(t, b/a) \right) \right] = \frac{1}{\lambda_0} + \frac{b}{a\lambda_1} + \inf_{t\le T} \varphi(t),$$

where $\varphi(t) \triangleq e^{-\lambda_0 t}\left[-(1/\lambda_0) - b\right] + e^{-\lambda_1 t}\left[-(b/(a\lambda_1)) + b\right]$. If $(1/a) + (1/b) < \lambda_1 - \lambda_0$, then $\varphi'(0) = 1 + \lambda_0 b + (b/a) - b\lambda_1 < 0$. Hence $\varphi(\cdot)$ is strictly decreasing in the neighborhood of $t = 0$, and $V(b/a) < (1/\lambda_0) + (b/(a\lambda_1)) + \varphi(0) = 0 - h(b/a)$. Therefore, $b/a \in \mathbf{C}_1 \subseteq \mathbf{C}_n \subseteq \mathbf{C}$ for every $n \ge 1$ are not empty. □

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