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Optimal Multiple-Stopping of Linear Diffusions

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Motivated by the analysis of financial instruments with multiple exercise rights of American type and mean reverting underlyers, we formulate and solve the optimal multiple-stopping problem for a general linear regular diffusion process and a general reward function. Instead of relying on specific properties of geometric Brownian motion and call and put option payoffs like in most of the existing literature, we use general theory of optimal stopping for diffusions, and we illustrate the resulting optimal exercise policies by concrete examples and constructive recipes.

Key words: Optimal multiple stopping; Snell envelope; diffusions; swing options

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1. Introduction. The purpose of this paper is to contribute to the mathematical theory of optimal multiple stopping, as motivated by the analysis of financial options with multiple exercises of the American type. It is surprising that despite a simple and intuitively natural formulation, this problem did not attract in the probability literature the attention it deserves. Instruments with multiple American exercises are ubiquitous in financial engineering. We find them in the design and analysis of executive stock option programs (see, for example, Sircar and Xiong [31], Leung and Sircar [24] and the references therein), in the indentures of many over-the-counter exotic fixed income markets instruments (see, for example, Meinshausen and Hambly [28] for a Monte Carlo analysis of multiple chooser swaps), or in the energy markets (see, for example, Jaillet et al. [17] for the numerical analysis of energy swing contracts and Carmona and Touzi [11] for their mathematical analysis in the case of geometric Brownian motion).

In this paper, we investigate the multiple optimal stopping problem for *general linear regular diffusion processes*. Even if geometric Brownian motion can be viewed as an appropriate model for some applications (e.g., executive stock option programs), it fails to capture important characteristic features of interest rates and commodities time series, mean-reversion being the most obvious. The interested reader is referred to Schwartz [29], Jaillet, Ronn, and Tompaidis [17], Barlow [3], Dixit and Pindyck [14] for the examples. Even if mean-reversion is only documented for the historical statistics of the underlyers, all the pricing (i.e., risk-neutral) models used by financial engineers account for this property. Therefore, pricing multi-exercise American-type options under diffusion models beyond geometric Brownian motion is important. Finally, we stress the fact that our analysis is not limited to the case of the *hockey-stick* payoff functions of the call and put options, as it handles general payoff functions.

First, we show as in Carmona and Touzi [11] that, by introducing appropriate Snell envelopes, the optimal multiple-stopping problem can be reduced to a sequence of ordinary optimal stopping problems that can be solved iteratively. Our result here is, however, stronger than theirs in several directions. Carmona and Touzi [11] show it when (i) the payoff process has a.s. continuous sample paths, (ii) its supremum over the entire horizon has some finite high-order moment, and (iii) it is adapted to a left-continuous filtration all of whose martingales must also have continuous sample paths. They impose those conditions in order to make sure that Snell envelopes have a.s. continuous sample paths and are left-continuous in expectation. Their first and third conditions disallow jump processes, which we also come across in the literature as proper models for the underlyers in pricing certain financial options in energy markets. Moreover, their second condition excludes general payoff functions that are encountered in the ever-expanding world of complex compounded financial and real options. Since Carmona and Touzi [11] focus exclusively on pricing multiple-exercise put options (namely, options with bounded terminal payoff functions), this compromise in their treatment of general optimal multiple-stopping problem is suitable for their purpose and allows them to avoid technical difficulties, which they call "beyond the scope of [their] paper". In this paper, one of our purposes is to price multiple-exercise options with general payoff processes, and we are able to prove the key result, namely the reduction of general optimal multiple-stopping problems to a sequence of ordinary optimal stopping problems, for payoff processes (i) with càdlàg (right-continuous with left-limits) sample paths, (ii) without any conditions on the moments of the supremum of the payoff process, and (iii) adapted to any filtration satisfying the usual conditions.

Carmona and Touzi [11] show this key reduction result primarily for the infinite-horizon problems, which immediately extends to finite-horizon problems after the following simple observation: every finite-horizon problem may be turned into an infinite horizon problem by simply setting the value of the payoff process identically to zero after the maturity. Here we also limit the discussion to infinite-horizon optimal multiple-stopping problems, but the same results extend to finite-horizon problems in the same trivial way. Now that we prove here the key reduction result under more general conditions as described above, their numerical algorithms for finite horizon problems, based on time discretization and successive runs of backward-dynamic-programming iterations, is unleashed from the restraints put on the payoff processes by those authors only to prove the same key reduction result, and this provides the theoretical justification for applying the same numerical algorithms to finite-horizon problems with much more general payoff processes.

The remainder of Carmona and Touzi's [11] work is exclusively on pricing infinite-horizon multiple-exercise put option for geometric Brownian motion, whereas we deal in this paper with infinite-horizon multiple-exercise options with general payoff functions for general regular linear diffusions. The boundedness of the put option's terminal payoff function allows them to use the basic reduction result; as argued above, the limitations of their result limit the application of the same idea to more general (e.g., unbounded) payoff functions—even when the underlyer is a geometric Brownian motion. Since the put option's payoff function is also decreasing and vanishes for large values of the underlyer, the optimal strategy is rather obvious (namely, exercise every right—whenever it is allowed—as soon as the underlyer is found below a suitable threshold). Therefore, the majority of their remaining work is to verify the correctness of this guess. For more general payoff functions, it is often difficult to even guess an optimal exercise strategy, as illustrated by some examples in Dayanik and Karatzas [13] and Dayanik [12]. Verification of a good guess is also very demanding in general; very popular variational formulation, also hinted at by Carmona and Touzi [11], typically writes down a series of free-boundary second-order ordinary differential equations (one for each disconnected continuation region), tries to solve them simultaneously, and use various techniques to show that one of the solutions indeed coincides with the value function of the optimal stopping problem. This procedure requires that one pays close attention to each special feature of the underlying problem and that special skills and tools (such as viscosity solutions of differential equations) be used, because the variational methods do not offer constructive algorithms. Carmona and Touzi [11] manage to avoid this burden of variational formulation and verify the correctness of their guess rather easily, thanks to very explicit formulas for the Laplace transforms of (geometric) Brownian motion's one-sided exit times. For general payoff functions and/or diffusion processes, those advantages disappear, and one cannot unfortunately go far enough with variational methods, either.

Instead, in this paper we use constructive potential-theoretic solution methods developed by Dayanik and Karatzas [13] and Dayanik [12] for optimal stopping of linear diffusions. We show how to *construct* the value function of an optimal multiple-stopping problem with general payoff function and general underlying linear diffusion. We describe when optimal multiple-stopping strategies exist and how to find them. We illustrate the methods on several examples. We show that exercise boundaries of perpetual call and put options are given by a sequence of points. We analyze several explicit diffusion models for which we give algorithmic constructions of these exercise boundaries.

The reduction of optimal multiple-stopping problem to a sequence of ordinary optimal stopping problems reminds similar approaches implemented in the literature to reduce optimal singular/impulse control and switching problems to a sequence of optimal stopping problems; see, for example, Karatzas and Shreve [18, 19], Boetius and Kohlmann [5], Boetius [4], Yushkevich [32, 33], Cairoli and Dalang [8, Chapter 10], Mandelbaum and Vanderbei [26, 27], Mandelbaum, Shepp, and Vanderbei [25], Carmona and Ludkovski [9, 10]. The common trait of our paper and these works is in the reinterpretation of dynamic programming equation, which may be described in the following general terms. For a general stochastic control problem with finite number of control actions, one can identify multiple-stopping problems after writing down the dynamic-programming equation. The state space of the controlled process can often be divided into disjoint subsets in which taking a specific control action is optimal. Then the original control problem may be seen as a sequence of optimal stopping problems, which determine switching times between different control actions.

We close this introduction summarizing the content of the paper. First we give an overview of the infinite-horizon optimal stopping problem for general continuous-parameter processes in Section 2. In Section 3, we formulate the optimal multiple-stopping problem for general continuous-parameter processes and show that it can be reduced to a sequence of ordinary optimal stopping problems. Then we specialize to standard Markov processes in Section 4 and describe the solution in terms of excessive functions. We revisit the same problem for one-dimensional time-homogeneous diffusions in Section 5 and illustrate the methods on examples in Section 6.

2. Optimal stopping theory: a short review. As we introduce the notation used throughout the paper, we summarize the main results of Karatzas and Shreve [21, Appendix D] on optimal stopping for a continuous-parameter process. Let $\{Y(t), \mathcal{F}_t;\, 0 \le t \le T\}$ be a nonnegative process with right-continuous paths and $Y(T) \le \overline{\lim}_{t\uparrow T} Y(t)$ a.s., defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and adapted to a filtration $\mathbb{F} \triangleq \{\mathcal{F}_t\}_{0\le t\le T}$ that satisfies the usual conditions. We shall assume that $\mathcal{F}(0)$ contains the sets of probability zero or one. The time horizon $T \in (0, +\infty]$ is a constant. If $T = +\infty$, then $\mathcal{F}_\infty \triangleq \sigma(\cup_{0\le t<+\infty}\mathcal{F}_t)$ and $Y(+\infty) \triangleq \overline{\lim}_{t\to+\infty} Y(t)$. Let $\mathcal{S}$ be the collection of $\mathbb{F}$–stopping times with values in $[0, T]$, and $\mathcal{S}_\sigma \triangleq \{\tau \in \mathcal{S};\, \tau \ge \sigma\}$ for every $\sigma \in \mathcal{S}$. The classical optimal stopping problem is to compute

$$Z_1(0) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}\{Y(\tau)\}$$

and to find $\tau^* \in \mathcal{S}$ at which the above supremum is attained, if such a stopping time exists. For each stopping time $\nu \in \mathcal{S}$ we introduce the random variable:

$$Z_1(\nu) \triangleq \operatorname{ess\,sup}_{\tau\in\mathcal{S}_\nu} \mathbb{E}\{Y(\tau)|\mathcal{F}_\nu\}.$$

Under the assumption that $Z_1(0)$ is finite, the following results hold:

Proposition 2.1 *The process $\{Z_1(t);\, 0 \le t \le T\}$ has a modification $\{Z_1^r(t);\, 0 \le t \le T\}$ that is a supermartingale with* càdlàg *paths. Moreover, $Z_1^r(\tau) = Z_1(\tau)$ a.s. for every $\tau \in \mathcal{S}$.*

Let $\{X_i(t);\, 0 \le t \le T\}$, $i = 1, 2$ be two arbitrary processes. One says that the process X_1 *dominates* the process X_2 if $\mathbb{P}\{X_1(t) \ge X_2(t)\,\text{for every}\, 0 \le t \le T\} = 1$. This notion is needed to guarantee the uniqueness of the process $Z_1^r(\cdot)$ identified in Proposition 2.1; see Proposition 2.2 below. It is called *the Snell envelope* of $\{Y(t);\, 0 \le t \le T\}$.

Proposition 2.2 *The process $Z_1^r(\cdot)$ dominates $Y(\cdot)$, and if $X(\cdot)$ is another* càdlàg *supermartingale dominating $Y(\cdot)$, then $X(\cdot)$ also dominates $Z_1^r(\cdot)$.*

Lemma 2.1 *Let $\sigma \in \mathcal{S}$ and $(\sigma_k)_{k\ge1} \subset \mathcal{S}_\sigma$ be such that $\sigma_k \downarrow \sigma$ almost surely. Then*

$$\int_A Z_1^r(\sigma)d\mathbb{P} = \lim_{k\to\infty} \int_A Z_1^r(\sigma_k)d\mathbb{P} \quad a.s., \quad A \in \mathcal{F}_\sigma.$$

Proof. Fix any $A \in \mathcal{F}_\sigma$. Since $Z_1^r(\cdot)$ has right-continuous paths, Fatou's lemma implies $\int_A Z_1^r(\sigma)d\mathbb{P} \le \underline{\lim} \int_A Z_1^r(\sigma_k)d\mathbb{P}$. On the other hand, $\int_A Z_1^r(\sigma)d\mathbb{P} \ge \overline{\lim} \int_A Z_1^r(\sigma_k)d\mathbb{P}$ by optional sampling, since $Z^r(\cdot)$ is a nonnegative $\mathbb{F}$–supermartingale. □

Proposition 2.3 *A stopping time τ^* is optimal if and only if (i) $Z_1^r(\tau^*) = Y(\tau^*)$ a.s. and (ii) the stopped supermartingale $\{Z^r(t \wedge \tau^*);\, 0 \le t \le T\}$ is a martingale.*

3. Multiple-stopping problem. In the remainder we introduce and study perpetual optimal multiple-stopping problems; namely, we set $T = +\infty$. Let $\delta > 0$ be a given constant and let us define $\mathcal{S}^n_\sigma \triangleq \{(\tau_1, \ldots, \tau_n);\, \tau_1 \in \mathcal{S}_\sigma;\, \tau_i \in \mathcal{S}_{\tau_{i-1}+\delta}, i = 2, \ldots, n\}$, $n \geq 1$ for every stopping time $\sigma \in \mathcal{S}$, and

$$Z_n(0) \triangleq \sup_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}^n_0} \mathbb{E}[Y(\tau_1) + \cdots + Y(\tau_n)], \qquad n \geq 1. \tag{1}$$

The number $Z_n(0)$ is the maximum expected payoff of a multiple-stopping option if it gives to the holder $n \geq 1$ rights to mark the underlying reward process, and if the holder is not allowed to mark more than once within any time-window of size less than δ. The constant $\delta > 0$ is sometimes called a "refracting time". For example in swing options, a refracting time is the minimum time a seller needs in order to fulfill an unscheduled delivery of additional commodity and is usually determined by the technological constraints on the production facilities or transmission networks; see, for example, Jaillet, Ronn, and Tompaidis [17], Carmona and Touzi [11].

The optimal multiple-stopping problem is to find the maximum expected reward $Z_n(0)$, and an optimal exercise strategy $(\tau_1, \ldots, \tau_n) \in \mathcal{S}^n_0$ that attains the supremum in (1), if one exists. We shall show that $Z_n(0)$ can be calculated by solving n optimal stopping problems sequentially. Let us introduce

$$Z_0(\sigma) \equiv 0, \quad \text{and} \quad Z_n(\sigma) \triangleq \operatorname{ess\,sup}_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}^n_\sigma} \mathbb{E}\Big\{ \sum_{i=1}^n Y(\tau_i) \Big| \mathcal{F}_\sigma \Big\}, \quad \sigma \in \mathcal{S}. \tag{2}$$

We will assume that $Z_1(0)$ is finite. Since, as it is easily seen, $Z_n(0) \leq nZ_1(0)$, every $Z_n(0)$, $n \geq 1$ will also be finite.

Lemma 3.1 *For every $\sigma \in \mathcal{S}$, the family $\mathcal{T} \triangleq \big\{\mathbb{E}[\sum_{i=1}^n Y(\tau_i)|\mathcal{F}_\sigma];\, (\tau_1, \ldots, \tau_n) \in \mathcal{S}^n_\sigma\big\}$ is directed upwards, and there exists a sequence $\{(\tau^k_1, \ldots, \tau^k_n)\}_{k\geq1} \subset \mathcal{S}^n_\sigma$ such that $Z_n(\sigma) = \lim_{k\to\infty} \uparrow \mathbb{E}[\sum_{i=1}^n Y(\tau^k_i)|\mathcal{F}_\sigma]$ almost surely.*

Proof. For $(\tau_1, \ldots, \tau_n)$ and $(\sigma_1, \ldots, \sigma_n)$ in $\mathcal{S}^n_\sigma$, define the event $A \triangleq \big\{\mathbb{E}[\sum_{i=1}^n Y(\tau_i)|\mathcal{F}_\sigma] \geq \mathbb{E}[\sum_{i=1}^n Y(\sigma_i)|\mathcal{F}_\sigma]\big\}$, and the stopping times $\nu_i \triangleq \tau_i 1_A + \sigma_i 1_{\Omega\setminus A}$, $i = 1, \ldots, n$. Then $(\nu_1, \ldots, \nu_n) \in \mathcal{S}^n_\sigma$, and

$$\mathbb{E}\Big[\sum_{i=1}^n Y(\nu_i)|\mathcal{F}_\sigma\Big] = \max\left(\mathbb{E}\Big[\sum_{i=1}^n Y(\tau_i)|\mathcal{F}_\sigma\Big], \mathbb{E}\Big[\sum_{i=1}^n Y(\sigma_i)|\mathcal{F}_\sigma\Big]\right).$$

Hence, $\mathcal{T}$ is directed upwards, and the second part follows from the properties of an essential supremum; see, e.g., Karatzas and Shreve [21, Appendix A]. □

Lemma 3.2 *If $n \geq 0$, $\tau \in \mathcal{S}$, and $\sigma \in \mathcal{S}_\tau$, then $\mathbb{E}\{Z_n(\sigma)|\mathcal{F}_\tau\} \leq Z_n(\tau)$ almost surely.*

Proof. Let $\{(\tau^k_1, \ldots, \tau^k_n)\}_{k\geq1} \subset \mathcal{S}^n_\sigma$ be as in Lemma 3.1. Then we have $\mathbb{E}\{Z_n(\sigma)|\mathcal{F}_\tau\} = \lim_{k\to\infty} \mathbb{E}\big\{\sum_{i=1}^n Y(\tau^k_i)\big|\mathcal{F}_\tau\big\} \leq Z_n(\tau)$ almost surely by the monotone convergence theorem for conditional expectations. □

Proposition 3.1 *For every $\sigma \in \mathcal{S}$ and $n \geq 0$, we have*

$$Z_{n+1}(\sigma) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_\sigma} \mathbb{E}\Big\{Y(\tau) + \mathbb{E}[Z_n(\tau+\delta)|\mathcal{F}_\tau]\Big|\mathcal{F}_\sigma\Big\} \quad a.s. \tag{3}$$

Proof. Fix $\tau_1 \in \mathcal{S}_\sigma$. By Lemma 3.1, there exists a sequence $\{(\tau^k_2, \ldots, \tau^k_{n+1})\}_{k\geq1}$ in $\mathcal{S}^n_{\tau_1+\delta}$ such that $Z_n(\tau_1 + \delta) = \lim_{k\to\infty} \uparrow \mathbb{E}\{\sum_{i=2}^{n+1} Y(\tau^k_i)|\mathcal{F}(\tau_1 + \delta)\}$ almost surely. For every $k \geq 1$, we have $(\tau_1, \tau^k_2, \ldots, \tau^k_{n+1}) \in \mathcal{S}^{n+1}_\sigma$, and by the monotone convergence theorem,

$$\begin{aligned}
Z_{n+1}(\sigma) &\geq \varlimsup_{k\to\infty} \mathbb{E}\Big\{Y(\tau_1) + \sum_{i=2}^{n+1} Y(\tau^k_i)\Big|\mathcal{F}_\sigma\Big\} \\
&= \mathbb{E}\Big\{Y(\tau_1) + \lim_{k\to\infty} \mathbb{E}\big\{\sum_{i=2}^{n+1} Y(\tau^k_i)\big|\mathcal{F}(\tau_1+\delta)\big\}\Big|\mathcal{F}_\sigma\Big\} = \mathbb{E}\Big\{Y(\tau_1) + Z_n(\tau_1+\delta)\Big|\mathcal{F}_\sigma\Big\} \\
&= \mathbb{E}\Big\{Y(\tau_1) + \mathbb{E}[Z_n(\tau_1+\delta)|\mathcal{F}(\tau_1)]\Big|\mathcal{F}_\sigma\Big\}.
\end{aligned}$$

Since $\tau_1 \in \mathcal{S}_\sigma$ is arbitrary, this implies that $Z_{n+1}(\sigma)$ is greater than or equal to the right-hand side of (3) almost surely. On the other hand, for every $(\tau_1, \ldots, \tau_{n+1}) \in \mathcal{S}_\sigma^{n+1}$, we have $\tau_1 \in \mathcal{S}_\sigma$ and $(\tau_2, \ldots, \tau_{n+1}) \in \mathcal{S}^n_{\tau_1+\delta}$, and

$$\begin{aligned}\mathbb{E}\Big\{Y(\tau_1) + \sum_{i=2}^{n+1} Y(\tau_i)\Big|\mathcal{F}_\sigma\Big\} &= \mathbb{E}\Big\{Y(\tau_1) + \mathbb{E}\big\{\sum_{i=2}^{n+1} Y(\tau_i)\big|\mathcal{F}_{\tau_1+\delta}\big\}\Big|\mathcal{F}_\sigma\Big\} \\ &\le \mathbb{E}\Big\{Y(\tau_1) + Z_n(\tau_1+\delta)\Big|\mathcal{F}_\sigma\Big\} = \mathbb{E}\Big\{Y(\tau_1) + \mathbb{E}\{Z_n(\tau_1+\delta)|\mathcal{F}_{\tau_1}\}\Big|\mathcal{F}_\sigma\Big\} \\ &\le \operatorname{ess\,sup}_{\tau\in\mathcal{S}_\sigma} \mathbb{E}\Big\{Y(\tau) + \mathbb{E}[Z_n(\tau+\delta)|\mathcal{F}_\tau]\Big|\mathcal{F}_\sigma\Big\},\end{aligned}$$

which proves the opposite inequality. □

Let us now introduce the random variables

$$\overline{Z}_n(t) \triangleq \mathbb{E}\left[Z_n(t+\delta)|\mathcal{F}_t\right], \quad t \ge 0,\ n \ge 0. \tag{4}$$

Suppose that, for some $k \ge 0$, $\{\overline{Z}_k(t);\, t \ge 0\}$ has an adapted *càdlàg* modification $\overline{Z}^r_k(\cdot)$, and that $\mathbb{E}\{Z_k(\tau+\delta)|\mathcal{F}_\tau\} = \overline{Z}^r_k(\tau)$ a.s. for every $\tau \in \mathcal{S}$. Then it follows from Proposition 3.1 that $Z_{k+1}(\sigma) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_\sigma} \mathbb{E}\Big\{Y(\tau) + \overline{Z}^r_k(\tau)\Big|\mathcal{F}_\sigma\Big\}$ a.s., or

$$Z_{k+1}(\sigma) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_\sigma} \mathbb{E}\Big\{Y_{k+1}(\tau)\Big|\mathcal{F}_\sigma\Big\} \text{ a.s.}, \quad \sigma \in \mathcal{S}, \tag{5}$$

where the payoff process is $Y_{k+1}(t) \triangleq Y(t) + \overline{Z}^r_k(t)$, $t \ge 0$. Since $Y_{k+1}(\cdot)$ is a nonnegative $\mathbb{F}$-adapted process with right-continuous sample paths, Proposition 2.1 implies that the *Snell envelope* $Z^r_{k+1}(\cdot)$ of the process $\{Y_{k+1}(t);\, t \ge 0\}$ exists, and $Z_{k+1}(\sigma) = Z^r_{k+1}(\sigma)$ a.s. for every $\sigma \in \mathcal{S}$.

Using the Snell envelope $Z^r_{k+1}(\cdot)$, we can show that $\overline{Z}_{k+1}(\cdot)$ has an adapted *càdlàg* modification $\overline{Z}^r_{k+1}(\cdot)$ such that $\mathbb{E}\{Z_{k+1}(\tau+\delta)|\mathcal{F}_\tau\} = \overline{Z}^r_{k+1}(\tau)$ a.s. for every $\tau \in \mathcal{S}$. We then proceed in the same manner as before. In the meantime, since $Z_0(\cdot) \equiv 0$ is itself the Snell envelope of $Y_0(\cdot) \equiv 0$, we can take $k = 0$ at the beginning of the previous paragraph and characterize $Z_n(\cdot)$ for every $n \ge 0$ in terms of the Snell envelopes of a sequence of reward processes.

Lemma 3.3 *The process $\{\overline{Z}_n(t), \mathcal{F}_t;\, t \ge 0\}$, $n \ge 0$ of (4) is a supermartingale.*

Proof. For $0 \le s \le t$, $\mathbb{E}\{\overline{Z}_n(t)|\mathcal{F}_s\} = \mathbb{E}\{Z_n(t+\delta)|\mathcal{F}_s\} = \mathbb{E}\big\{\mathbb{E}\{Z_n(t+\delta)|\mathcal{F}_{s+\delta}\}\big|\mathcal{F}_s\big\} \le \mathbb{E}\{Z_n(s+\delta)|\mathcal{F}_s\} = \overline{Z}_n(s)$ by Lemma 3.2. □

Proposition 3.2 *For every $n \ge 0$, $\overline{Z}_n(\cdot)$ of (4) has an adapted* càdlàg *modification $\overline{Z}^r_n(\cdot)$, and $\mathbb{E}\{Z_n(\tau+\delta)|\mathcal{F}_\tau\} = \overline{Z}^r_n(\tau)$ a.s. for every $\tau \in \mathcal{S}$. Furthermore,*

$$Z_{n+1}(\sigma) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_\sigma} \mathbb{E}\Big\{Y_{n+1}(\tau)\Big|\mathcal{F}_\sigma\Big\} \text{ a.s.} \quad \forall \sigma \in \mathcal{S}, \tag{6}$$

where $Y_{n+1}(t) \triangleq Y(t) + \overline{Z}^r_n(t)$, $t \ge 0$ is an $\mathbb{F}$-adapted càdlàg *process.*

Remark 3.1 Proposition 2.1 implies that, for $n \ge 0$ the *Snell envelope* $Z^r_n(\cdot)$ of the process $\{Y_n(t);\, t \ge 0\}$ exists, and

$$Z_n(\sigma) = Z^r_n(\sigma) \quad \text{a.s.} \quad \forall\, \sigma \in \mathcal{S}. \tag{7}$$

Moreover, $\mathbb{E}\{Z^r_n(\tau+\delta)|\mathcal{F}_\tau\} = \overline{Z}^r_n(\tau)$ a.s. for every $\tau \in \mathcal{S}$.

Proof of Proposition 3.2. Since $Z_0(\cdot) \equiv 0$, we can take $\overline{Z}^r_0(\cdot) \equiv 0$. Moreover, $Y_1(t) = Y(t)$, $t \ge 0$ is $\mathbb{F}$-adapted with right-continuous sample paths, and the claims hold for $n = 0$.

Let us assume that the proposition holds for $n-1$ and prove it for n. By hypothesis, $\overline{Z}^r_{n-1}$ exists. Therefore, $Y_n(t) \triangleq Y(t) + \overline{Z}^r_{n-1}(t)$ is adapted to $\mathbb{F}$ and has right-continuous sample paths. By Remark 3.1, the Snell envelope $Z^r_n(\cdot)$ of $Y_n(\cdot)$ exists, and

$$\mathbb{E}\{\overline{Z}_n(t)\} = \mathbb{E}\{Z_n(t+\delta)\} = \mathbb{E}\{Z^r_n(t+\delta)\}, \quad t \ge 0. \tag{8}$$

For every $(t_k)_{k\geq 1} \subset \mathbb{R}$ such that $t_k \downarrow t$, let $\sigma \equiv t$ and $\sigma_k \equiv t_k + \delta$ and $A = \Omega$ in Lemma 2.1. From (8), we obtain $\lim_{k\to\infty} \mathbb{E}\{\overline{Z}_n(t_k)\} = \lim_{k\to\infty} \mathbb{E}\{Z_n^r(t_k+\delta)\} = \mathbb{E}\{Z_n^r(t+\delta)\} = \mathbb{E}\{\overline{Z}_n(t)\}$; namely, $t \mapsto \mathbb{E}\{\overline{Z}_n(t)\}$ is right-continuous. Since $\{\overline{Z}_n(t);\ t\geq 0\}$ is also a supermartingale by Lemma 3.3, $\overline{Z}_n(\cdot)$ has an $\mathbb{F}$-adapted *càdlàg* modification $\overline{Z}_n^r(\cdot)$; see Karatzas and Shreve [20, Theorem 3.13].

The process $\{\overline{Z}_n^r(t);\ t\geq 0\}$ is a supermartingale with *càdlàg* paths. Thus, for every $\sigma \in \mathcal{S}$ and for every sequence $(\sigma_k)_{k\geq 1} \subset \mathcal{S}_\sigma$ such that $\sigma_k \downarrow \sigma$, one can check as in Lemma 2.1 that

$$\int_A \overline{Z}_n^r(\sigma) d\mathbb{P} = \lim_{k\to\infty} \int_A \overline{Z}_n^r(\sigma_k) d\mathbb{P} \quad \text{a.s.}, \quad A \in \mathcal{F}_\sigma. \tag{9}$$

For every $k \geq 1$ and $t \in \mathbb{R}$, define $\delta_k(0) \triangleq 0$ and $\delta_k(t) \triangleq ki/2^k$ if $k(i-1)/2^k < t \leq ki/2^k$ for some $i \geq 1$. For every stopping time $\sigma \in \mathcal{S}$, $\sigma_k \triangleq \delta_k(\sigma)$ is a stopping time that takes countably many distinct values, and $\sigma_k \downarrow \sigma$ almost surely. Thus

$$\begin{aligned}\int_A \overline{Z}_n^r(\sigma_k) d\mathbb{P} &= \int_A \mathbb{E}\{Z_n(\sigma_k+\delta)|\mathcal{F}_{\sigma_k}\} d\mathbb{P} \\ &= \int_A Z_n(\sigma_k+\delta) d\mathbb{P} = \int_A Z_n^r(\sigma_k+\delta) d\mathbb{P} \quad \text{a.s.}, \quad A \in \mathcal{F}_\sigma, \end{aligned} \tag{10}$$

since $\mathbb{P}\big(\overline{Z}_n^r(t) = \mathbb{E}\{Z_n(t+\delta)|\mathcal{F}_t\}\big) = 1$ for every $t \geq 0$ and (7) holds. By taking the limits of both sides in (10), we obtain

$$\int_A \overline{Z}_n^r(\sigma) d\mathbb{P} = \int_A Z_n^r(\sigma+\delta) d\mathbb{P} = \int_A Z_n(\sigma+\delta) d\mathbb{P} \quad \text{a.s.}, \quad A \in \mathcal{F}_\sigma, \tag{11}$$

following from (9), Lemma 2.1, and (7). Finally, (11) implies $\overline{Z}_n^r(\sigma) = \mathbb{E}\{Z_n(\sigma+\delta)|\mathcal{F}_\sigma\}$ almost surely. The remainder follows from Proposition 3.1. □

4. Markovian case. Let $X = (X(t), \mathcal{F}_t, \mathbb{P}_x)$ be a standard Markov process on a semicompact state space $(E, \mathcal{E})$. Let $h : E \mapsto [0, +\infty)$ be a measurable C_0-continuous function; i.e., $\lim_{t\downarrow 0} h(X(t)) = h(X(0))$ a.s., and let β denote the risk free interest rate. The reward process $Y(t) \triangleq e^{-\beta t} h(X(t))$, $t \geq 0$ of the previous section is nonnegative $\mathbb{F}$-adapted and right-continuous, and the value functions V_n are defined on the state space by

$$V_n(x) \triangleq \sup_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}_0^n} \mathbb{E}_x\left\{\sum_{i=1}^n e^{-\beta\tau_i} h(X(\tau_i))\right\}, \quad x \in E,\ n \geq 1. \tag{12}$$

In this section, we characterize $V_n(\cdot)$ of (12) in terms of the β-excessive functions of the Markov process. Recall that a measurable function $f : E \mapsto (-\infty, +\infty]$ is said to be β-excessive for X, if for every $x \in E$

$$f(x) \geq \mathbb{E}_x\{e^{-\beta t} f(X(t))\},\ t \geq 0 \quad \text{and} \quad f(x) = \lim_{t\downarrow 0} \mathbb{E}_x\{e^{-\beta t} f(X(t))\}. \tag{13}$$

The following results are well-known; see, e.g., Shiryaev [30, pp.116–117] and Fakeev [15]:

- E.1. A nonnegative β-excessive function is C_0-continuous.
- E.2. If $f(\cdot)$ is a finite β-excessive function, then $e^{-\beta t} f(X(t))$, $t \geq 0$ is a *càdlàg* $\mathbb{F}$-adapted supermartingale.
- E.3. If $g : E \mapsto [0, +\infty)$ is measurable and C_0-continuous, then the smallest β-excessive majorant of $g(\cdot)$ exists.
- E.4. If $g(\cdot)$ is the same as in E.3, then $V(x) \triangleq \sup_{\tau\in\mathcal{S}} \mathbb{E}_x\{e^{-\beta\tau} g(X(\tau))\}$, $x \in E$ is the smallest β-excessive majorant of $g(\cdot)$. For every $t \geq 0$,
$$\operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} \mathbb{E}_x\big\{e^{-\beta\tau} g(X(\tau))\big|\mathcal{F}_t\big\} = e^{-\beta t} V(X(t)), \qquad \mathbb{P}_x\text{-a.s.}$$
If V is finite, then $\{e^{-\beta t} V(X(t))\}_{t\geq 0}$ is the Snell envelope of $\{e^{-\beta t} g(X(t))\}_{t\geq 0}$.

Proposition 4.1 *Suppose V_1 is finite. Let $V_0 \equiv 0$, and define for every $n \geq 1$*

$$g_n(x) \triangleq \mathbb{E}_x\big\{e^{-\beta\delta} V_n(X(\delta))\big\} \quad \text{and} \quad h_{n+1}(x) \triangleq h(x) + g_n(x), \quad x \in E. \tag{14}$$

Then V_n is the smallest β-excessive majorant of h_n for every $n \geq 1$, and for $t \geq 0$

$$\operatorname{ess\,sup}_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}_t^n} \mathbb{E}_x\left\{\textstyle\sum_{i=1}^n e^{-\beta\tau_i} h(X(\tau_i))\Big|\, \mathcal{F}_t\right\} = e^{-\beta t} V_n(X(t)), \quad \mathbb{P}_x\text{-a.s.} \tag{15}$$

Proof. The proposition is true for V_1 by E.4. We shall assume that it is true for n, and prove it for $n+1$ by using Proposition 3.2.

Let Z_n and $\overline{Z}_n$ be as in (2) and (4), respectively. By induction hypothesis and (15), we have $Z_n(t) = e^{-\beta t}V_n(X(t))$ a.s. for every $t \geq 0$. Therefore,

$$\overline{Z}_n(t) = \mathbb{E}_x\{Z_n(t+\delta)|\mathcal{F}_t\} = \mathbb{E}_x\{e^{-\beta(t+\delta)}V_n(X(t+\delta))|\mathcal{F}_t\} = e^{-\beta t}g_n(X(t)), \text{ a.s.}$$

for every $t \geq 0$. By Proposition 3.2, $V_{n+1}(x) = \sup_{\tau\in\mathcal{S}_0}\mathbb{E}_x\{Y_{n+1}(\tau)\}$, where

$$Y_{n+1}(t) = Y(t) + \overline{Z}_n(t) = e^{-\beta t}(h+g_n)(X(t)) \quad \text{a.s.}, \ t \geq 0. \tag{16}$$

If we can show that g_n is C_0-continuous, then $h+g_n$ will be a nonnegative C_0-continuous function, and its smallest β-excessive majorant will exist by E.3. Then E.4 will imply that V_{n+1} is the smallest β-excessive majorant of $h+g_n$, and

$$\begin{aligned} e^{-\beta t}V_{n+1}(X(t)) &= \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_x\{e^{-\beta\tau}(h+g_n)(X(\tau))|\mathcal{F}_t\} && \text{by E.4} \\ &= \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_x\{Y_{n+1}(\tau))|\mathcal{F}_t\} && \text{by (16)} \\ &= Z_{n+1}(t) && \text{by (6)} \\ &= \operatorname{ess\,sup}_{(\tau_1,\ldots,\tau_{n+1})\in\mathcal{S}_t^{n+1}}\mathbb{E}_x\left\{\sum_{i=1}^{n+1}e^{-\beta\tau_i}h(X(\tau_i))\,\middle|\,\mathcal{F}_t\right\} && \text{by (2)} \end{aligned}$$

$\mathbb{P}_x$-a.s. for every $t \geq 0$, which proves (15) for $n+1$.

We claim that g_n is nonnegative and β-excessive. The C_0-continuity of g_n will then follow from E.1. It is nonnegative since h, and therefore V_n, is nonnegative. Because $V_n \leq nV_1$ and V_1 is finite, V_n is finite. By induction hypothesis, V_n is a finite β-excessive function. By E.2, $e^{-\beta t}V_n(X(t))$ is a *càdlàg* supermartingale. Therefore, $t \mapsto \mathbb{E}_x\{e^{-\beta t}V_n(X(t))\}$ is right-continuous, and

$$\lim_{t\downarrow 0}\mathbb{E}_x\{e^{-\beta t}g_n(X(t))\} = \lim_{t\downarrow 0}\mathbb{E}_x\{e^{-\beta(t+\delta)}V_n(X(t+\delta))\} = g_n(x).$$

Finally, $\mathbb{E}_x\{e^{-\beta t}g_n(X(t))\} = \mathbb{E}_x\{e^{-\beta(t+\delta)}V_n(X(t+\delta))\} \leq \mathbb{E}_x\{e^{-\beta\delta}V_n(X(\delta))\} = g_n(x)$. Hence, g_n is β-excessive. □

5. The case of regular linear diffusions. In the sequel we suppose that the process X of Section 4 is a time-homogeneous regular linear diffusion with dynamics

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t), \quad t \geq 0,$$

where B is a standard Brownian motion on $\mathbb{R}$, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the augmentation of the natural filtration of X that satisfies the usual conditions. We shall assume that the state space of X is an interval $\mathcal{I} = (a,b)$ for some $-\infty \leq a < b \leq +\infty$, and that the boundaries a and b are natural (other boundary types can be handled similarly; see Dayanik and Karatzas [13] and Dayanik [12]). Let τ_y be the first hitting time of $y \in \mathcal{I}$ by X, and let $c \in I$ be a fixed point of the state space. For every $\beta \geq 0$ we set

$$\psi(x) \triangleq \left\{\begin{array}{ll} \mathbb{E}_x\{e^{-\beta\tau_c}1_{\{\tau_c<\infty\}}\}, & x \leq c \\ 1/\mathbb{E}_c\{e^{-\beta\tau_x}1_{\{\tau_x<\infty\}}\}, & x > c \end{array}\right\}, \quad \varphi(x) \triangleq \left\{\begin{array}{ll} 1/\mathbb{E}_c\{e^{-\beta\tau_x}1_{\{\tau_x<\infty\}}\}, & x \leq c \\ \mathbb{E}_x\{e^{-\beta\tau_c}1_{\{\tau_c<\infty\}}\}, & x > c \end{array}\right\}, \tag{17}$$

and

$$F(x) = \frac{\psi(x)}{\varphi(x)}, \quad x \in \mathcal{I}. \tag{18}$$

Then $F(\cdot)$ is continuous and strictly increasing, $F(a+) = 0$ and $F(b-) = +\infty$ for every $\beta > 0$; see, e.g., Itô and McKean [16], Karlin and Taylor [22]. In this section, we shall redefine

$$h(X(\tau)) = 0 \quad \text{on } \{\tau = +\infty\}. \tag{19}$$

If $h(\cdot)$ is the payoff function of an American-type option, then (19) implies that no payment is received unless the option is exercised. Therefore, (19) is more natural in finance applications than setting $h(X(\tau)) = \overline{\lim}_{t\to\infty}h(X(t))$ on $\{\tau = \infty\}$. However, the results of previous sections are still valid under (19), and β-excessive functions are easily characterized in terms of the functions F and φ.

Proposition 5.1 *A measurable function $U : \mathcal{I} \mapsto [0,+\infty)$ is β-excessive for X if and only if $(U/\varphi)\circ F^{-1}$ is concave on $[0,+\infty)$, for every $\beta \geq 0$.*

Proof. If U is nonnegative, finite, and β-excessive, then E.2 and optional sampling imply that $U(x) \geq \mathbb{E}_x\{e^{-\beta\tau}U(X(\tau))\}$ for all $\tau \in \mathcal{S}$ and $x \in \mathcal{I}$. Therefore, the concavity of $(U/\varphi)\circ F^{-1}$ follows from Dayanik and Karatzas [13, Proposition 5.9].

If $(U/\varphi)\circ F^{-1}$ is concave, then U is continuous, and $U(x) \geq \mathbb{E}_x\{e^{-\beta\tau}U(X(\tau))\}$ for every $\tau \in \mathcal{S}$ and $x \in \mathcal{I}$ by the same proposition cited above. Therefore, $e^{-\beta t}U(X(t))$ is a *càdlàg* supermartingale, and $t \mapsto \mathbb{E}_x\{e^{-\beta t}U(X(t))\}$ is right-continuous, and (13) follows. □

Proposition 5.2 *All the V_n's are finite if and only if*

$$\ell_a \triangleq \varlimsup_{x\downarrow a} \frac{h^+(x)}{\varphi(x)} < +\infty \quad \text{and} \quad \ell_b \triangleq \varlimsup_{x\uparrow b} \frac{h^+(x)}{\psi(x)} < +\infty. \tag{20}$$

Moreover, if (i) (20) holds, and (ii) h_n is as in Proposition 4.1, and (iii) W_n is the smallest nonnegative concave majorant of $H_n \triangleq (h_n/\varphi)\circ F^{-1}$ on $[0,+\infty)$, then

$$V_n(x) = \varphi(x)W_n(F(x)), \qquad x \in \mathcal{I},\ n \geq 1. \tag{21}$$

Proof. The finiteness of V_n follows from Proposition 5.10 and $V_1 \leq V_n \leq nV_1$, $n \geq 1$, and the rest from Proposition 5.12 in Dayanik and Karatzas [13]. □

In the remainder of this section, we assume that (20) holds. By Propositions 5.1 and 5.2, $V_n(\cdot)$ and $g_n(\cdot)$ are finite and continuous, and if

$$\mathbf{\Gamma}_n \triangleq \{x \in \mathcal{I} : V_n(x) = h_n(x)\} \quad \text{and} \quad \sigma_n \triangleq \inf\{t \geq 0 : X(t) \in \mathbf{\Gamma}_n\}, \tag{22}$$

then $\mathbf{\Gamma}_n$ is closed, and σ_n is a stopping time for every $n \geq 1$. By Proposition 4.1,

$$V_n(x) = \sup_{\tau\in\mathcal{S}} \mathbb{E}_x\{e^{-\beta\tau}h_n(X(\tau))\}, \quad x \in \mathcal{I}. \tag{23}$$

If (23) has an optimal stopping time, then σ_n is also optimal for the same problem. In fact, Dayanik and Karatzas [13, Proposition 5.13 and 5.14] showed the following:

Proposition 5.3 *Fix any $n \geq 1$. The stopping time σ_n of (22) is optimal for (23), if and only if either*

$$\ell_a^{(n)} \triangleq \varlimsup_{x\downarrow a} \frac{h_n^+(x)}{\varphi(x)} = 0 \quad \text{and} \quad \ell_b^{(n)} \triangleq \varlimsup_{x\uparrow b} \frac{h_n^+(x)}{\psi(x)} = 0, \tag{24}$$

or (i) if $\ell_a^{(n)} > 0$, then there is no $r \in \mathcal{I}$ such that $(a,r) \subseteq \mathcal{I}\backslash\mathbf{\Gamma}_n$, and (ii) if $\ell_b^{(n)} > 0$, then there is no $l \in \mathcal{I}$ such that $(l,b) \subseteq \mathcal{I}\backslash\mathbf{\Gamma}_n$.

Proposition 5.4 *Suppose that the stopping time σ_n of (22) is optimal for (23) for every $n = 1,\ldots,m$. Let $\tau_1^{(1)} \triangleq \sigma_1$, and introduce for every $n \geq 2$, the stopping times*

$$\tau_1^{(n)} \triangleq \sigma_n, \quad \text{and} \quad \tau_i^{(n)} \triangleq \tau_{i-1}^{(n)} + \delta + \sigma_{n-i+1}\circ\theta_{\tau_{i-1}^{(n)}+\delta}, \quad i = 2,\ldots,n, \tag{25}$$

where θ is the time-shift operator. Then the stopping strategy $(\tau_1^{(n)},\ldots,\tau_n^{(n)}) \in \mathcal{S}^n$ is optimal for the multiple-stopping problem (12) for every $n = 1,\ldots,m$.

Proof. We will prove the proposition by induction on n. For $n = 1$, $V_1(x) = \mathbb{E}_x\{e^{-\beta\sigma_1}h_1(X_{\sigma_1})\} = \mathbb{E}_x\{e^{-\beta\tau_1^{(1)}}h(X_{\tau_1^{(1)}})\}$ and $\tau_1^{(1)} \in \mathcal{S}^1$ is indeed optimal.

Let us assume that $(\tau_1^{(n)},\ldots,\tau_n^{(n)})$ is optimal for (12) for some $1 \leq n \leq m-1$ and prove the same for $n+1$. Since $\tau_1^{(n+1)} = \sigma_{n+1}$ is optimal for (23) for $n+1$,

$$\begin{aligned}
V_{n+1}(x) &= \mathbb{E}_x\{e^{-\beta\tau_1^{(n+1)}}h_{n+1}(X_{\tau_1^{(n+1)}})\} = \mathbb{E}_x\{e^{-\beta\tau_1^{(n+1)}}(h+g_n)(X_{\tau_1^{(n+1)}})\} \\
&= \mathbb{E}_x\{e^{-\beta\tau_1^{(n+1)}}h(X_{\tau_1^{(n+1)}})\} + \mathbb{E}_x\{e^{-\beta(\tau_1^{(n+1)}+\delta)}V_n(X_{\tau_1^{(n+1)}+\delta})\} \\
&= \mathbb{E}_x\{e^{-\beta\tau_1^{(n+1)}}h(X_{\tau_1^{(n+1)}})\} + \mathbb{E}_x\big\{e^{-\beta(\tau_1^{(n+1)}+\delta)}\mathbb{E}_{X_{\tau_1^{(n+1)}+\delta}}\textstyle\sum_{i=1}^n e^{-\beta\tau_i^{(n)}}h(X_{\tau_i^{(n)}})\big\} \\
&= \mathbb{E}_x\{e^{-\beta\tau_1^{(n+1)}}h(X_{\tau_1^{(n+1)}})\} + \mathbb{E}_x\big\{\textstyle\sum_{i=1}^n e^{-\beta\rho_i^{(n+1)}}h(X_{\rho_i^{(n+1)}})\big\},
\end{aligned}$$

where $\rho_i^{(n+1)} \triangleq \tau_1^{(n+1)} + \delta + \tau_i^{(n)} \circ \theta_{\tau_1^{(n+1)}+\delta}$, $i = 1, \ldots, n$. The proof of the induction step will follow once we show that

$$\rho_i^{(n+1)} = \tau_{i+1}^{(n+1)}, \qquad i = 1, \ldots, n. \tag{26}$$

Note that $\rho_1^{(n+1)} \triangleq \tau_1^{(n+1)} + \delta + \sigma_n \circ \theta_{\tau_1^{(n+1)}+\delta} = \tau_2^{(n+1)}$, and (26) holds for $i = 1$. Suppose (26) is true for $1 \le i \le n-1$, and prove the same for $i+1$. We have

$$\begin{aligned} \rho_{i+1}^{(n+1)} &\triangleq \tau_1^{(n+1)} + \delta + \tau_{i+1}^{(n)} \circ \theta_{\tau_1^{(n+1)}+\delta} \\ &= \tau_1^{(n+1)} + \delta + \left(\tau_i^{(n)} + \delta + \sigma_{n-i} \circ \theta_{\tau_i^{(n)}+\delta}\right) \circ \theta_{\tau_1^{(n+1)}+\delta} \end{aligned} \tag{27}$$

by the definition (25) of $\tau_{i+1}^{(n)}$. Since $\rho_i^{(n+1)} \triangleq \tau_1^{(n+1)} + \delta + \tau_i^{(n)} \circ \theta_{\tau_1^{(n+1)}+\delta} = \tau_{i+1}^{(n+1)}$ by the induction hypothesis, it follows

$$(\tau_i^{(n)} + \delta) \circ \theta_{\tau_1^{(n+1)}+\delta} = \tau_i^{(n)} \circ \theta_{\tau_1^{(n+1)}+\delta} + \delta = \tau_{i+1}^{(n+1)} - \tau_1^{(n+1)},$$

and, since $(Y \circ \theta_\sigma) \circ \theta_\tau = Y \circ \theta_{\tau+\sigma\circ\theta_\tau}$ for every random variable Y and stopping times σ and τ, we have

$$\left(\sigma_{n-i} \circ \theta_{\tau_i^{(n)}+\delta}\right) \circ \theta_{\tau_1^{(n+1)}+\delta} = \sigma_{n-i} \circ \theta_{\tau_{i+1}^{(n+1)}+\delta}.$$

By plugging the last two equalities back into (27), we obtain

$$\begin{aligned} \rho_{i+1}^{(n+1)} &= \tau_1^{(n+1)} + \delta + \left(\tau_{i+1}^{(n+1)} - \tau_1^{(n+1)} + \sigma_{n-i} \circ \theta_{\tau_{i+1}^{(n+1)}+\delta}\right) \\ &= \tau_{i+1}^{(n+1)} + \delta + \sigma_{n-i} \circ \theta_{\tau_{i+1}^{(n+1)}+\delta} = \tau_{i+2}^{(n+1)} \end{aligned}$$

which completes the proof of both induction hypotheses. □

Corollary 5.1 *If both ℓ_a and ℓ_b of (20) are zero, then (24) holds, and $(\tau_1^{(n)}, \ldots, \tau_n^{(n)})$ is optimal for the multiple-stopping problem (12) for every $n \ge 1$.*

Proof. If we establish (24), the rest follows from Propositions 5.3 and 5.4. Since $h \le h_n = h + g_{n-1} \le h + V_{n-1} \le (n+1)V_1$, and the mapping $x \mapsto x^+$ is increasing $h^+ \le h_n^+ \le (n+1)V_1$.

Dayanik and Karatzas [13, Proposition 5.10] prove that $\lim_{x\downarrow a}(V_1/\varphi)(x) = \ell_a$ and $\lim_{x\uparrow b}(V_1/\psi)(x) = \ell_b$. From the previous inequalities, it follows $\ell_a \le \ell_a^{(n)} = \overline{\lim}_{x\downarrow a} \le (n+1)\ell_a$ and $\ell_b \le \ell_b^{(n)} = \overline{\lim}_{x\uparrow b} \le (n+1)\ell_b$. Since $\ell_a = \ell_b = 0$, (24) follows. □

6. Examples.

This final section is devoted to a detailed analysis of a set of natural examples for which explicit computations can be performed.

6.1 Brownian motion.

Let X be one-dimensional standard Brownian motion on $\mathcal{I} = \mathbb{R}$, the reward function be $h(x) \triangleq x^+$ for $x \in \mathcal{I}$, and fix $\beta > 0$.

The functions $\psi(\cdot)$ and $\varphi(\cdot)$ of (17) are the unique (up to a scalar multiple) increasing and decreasing solutions of $(1/2)u'' = \beta u$, respectively. We take $\psi(x) = e^{x\sqrt{2\beta}}$ and $\varphi(x) = e^{-x\sqrt{2\beta}}$, $x \in \mathbb{R}$ so that

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = e^{2x\sqrt{2\beta}}, \qquad x \in \mathbb{R}.$$

The boundaries $\pm\infty$ are natural, and $F(-\infty) = 0$ and $F(+\infty) = +\infty$. Clearly, $\ell_{-\infty}$ and $\ell_{+\infty}$ of (20) are zero. Therefore, all the V_n's of (12) are finite by Proposition 5.2, and the multiple stopping strategies $(\tau_1^{(n)}, \ldots, \tau_n^{(n)})$ of (25) are optimal by Corollary 5.1. Hence, the optimal multiple-stopping problem (12) reduces to the optimal stopping problem (23).

6.1.1 ($n = 1$).

By Proposition 5.2, we have $V_1(x) = \varphi W_1(F(x))$, $x \in \mathbb{R}$, where $W_1(\cdot)$ is the smallest nonnegative concave majorant of

$$H_1(y) \triangleq (h_1/\varphi)(F^{-1}(y)) = \frac{(\ln y)^+\sqrt{y}}{2\sqrt{2\beta}}, \qquad y \in [0, +\infty),$$

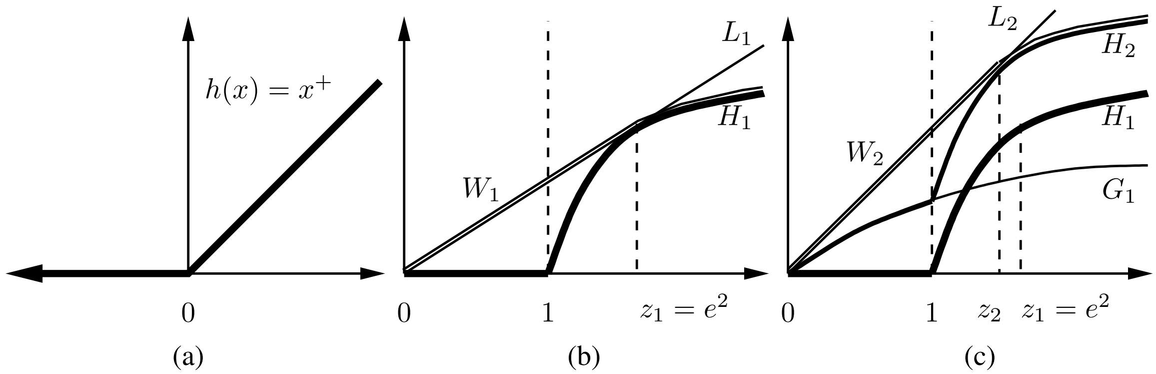

Figure 1: (Brownian motion). The sketches of (a) the reward function h, (b) the function H_1 and its smallest nonnegative concave majorant W_1, (c) $H_2 = H_1 + G_1$ and its smallest nonnegative concave majorant W_2.

which vanishes on $[0, 1)$ and is nonnegative strictly concave and increasing on $[1, +\infty)$; see Figure 1(b). Since $\lim_{y\to+\infty} H_1'(y) = 0$, there is a unique number $z_1 > 1$ such that

$$H_1(z_1)/z_1 = H_1'(z_1). \tag{28}$$

In fact $z_1 = e^2$, and $W_1(\cdot)$ coincides with $L_1(y) \triangleq yH_1'(z_1)$ on $[0, z_1]$ and with $H_1(\cdot)$ on $[z_1, +\infty)$. Now, let $x_1 \triangleq F^{-1}(z_1) = 1/\sqrt{2\beta}$. Then

$$V_1(x) = \varphi(x)W_1(F(x)) = \begin{cases} e^{x\sqrt{2\beta}-1}/\sqrt{2\beta}, & x \le x_1, \\ x, & x \ge x_1. \end{cases}$$

Since $\mathbf{\Gamma}_1 = \{x \in \mathbb{R} : V_1(x) = h_1(x)\} = F^{-1}\big(\{y \ge 0 : W_1(y) = H_1(y)\}\big) = [x_1, +\infty)$, the optimal stopping time of (22) is $\sigma_1 = \inf\{t \ge 0 : X(t) \ge x_1\}$.

6.1.2 ($n = 2$). We start by first finding the smallest nonnegative concave majorant W_2 of $H_2 \triangleq (h_2/\varphi) \circ F^{-1}$, where $h_2 = h + g_1$ and $g_1(x) = \mathbb{E}_x\{e^{-\beta\delta}V_1(X(\delta))\}$, $x \in \mathbb{R}$.

If $G_1(y) \triangleq (g_1/\varphi)(F^{-1}(y))$ for every $y \ge 0$, then $H_2 = H_1 + G_1$. Since g_1 is nonnegative, finite, and β-excessive, the function G_1 is concave by Proposition 5.1. Because G_1 is also nonnegative, its concavity implies that the right-derivative of $G_1(y)$ is nonnegative everywhere (otherwise, $G_1 < 0$ on $[y_0, +\infty)$ for some $y_0 \ge 0$); therefore, G_1 is also nondecreasing. Finally, $0 \le G_1 \le W_1$ and $\lim_{y\downarrow 0} W_1(y) = \lim_{y\downarrow 0} H_1(y) = 0$. Thus, $\lim_{y\downarrow 0} G_1(y) = 0$.

As shown in Figure 1(c), H_2 is concave both on $[0, 1]$ and $[1, +\infty)$. Since G_1 and H_1 are concave on $[1, +\infty)$ and $G_1 \le H_1$, we must have $0 \le \lim_{y\to\infty} G_1'(y) \le \lim_{y\to\infty} H_1'(y)$ (otherwise, $G_1 > H_1$ on $[y_1, +\infty)$ for some $y_1 \ge 0$). Since the latter is zero, $\lim_{y\to\infty} G_1'(y) = 0$. Hence, $\lim_{y\to\infty} H_2'(y) = 0$, and there is unique $z_2 > 1$ such that

$$H_2(z_2)/z_2 = H_2'(z_2). \tag{29}$$

It is then clear, as also seen from Figure 1(c), that the smallest nonnegative concave majorant W_2 of H_2 is the same as the straight line $L_2(y) = yH_2'(z_2)$ on $[0, z_2]$, and the same as H_2 on $[z_2, +\infty)$. If we define $x_2 \triangleq F^{-1}(z_2)$, then

$$V_2(x) = \varphi(x)W_2(F(x)) = \begin{cases} h_2(x_2)e^{-(x_2-x)\sqrt{2\beta}}, & x \le x_2, \\ h_2(x), & x > x_2. \end{cases} \tag{30}$$

It is also easy to see that $\mathbf{\Gamma}_2 = [x_2, +\infty)$ and $\sigma_2 = \inf\{t \ge 0 : X(t) \ge x_2\}$.

Next we prove that $x_2 \le x_1$. Note that

$$\frac{d}{dy}\left(\frac{H_n(y)}{y}\right) = \frac{1}{y}\left(H_n'(y) - \frac{H_n(y)}{y}\right), \quad y > 1,\ n = 1, 2. \tag{31}$$

Since H_n are concave on $[1, +\infty)$, the right-hand side of (31) is positive (negative) for $1 < y < z_n$ ($y > z_n$) and equals zero at $y = z_n$, thanks to (28) and (29). Hence, z_n is the global maximum on $[1, +\infty)$ of $y \mapsto H_n(y)/y$, which is increasing (decreasing) on $[1, z_n]$ ($[z_n, +\infty)$) for $n = 1, 2$. We have

$$\frac{H_2(z_1)}{z_1} = \frac{H_1(z_1)}{z_1} + \frac{G_1(z_1)}{z_1} = H_1'(z_1) + \frac{G_1(z_1)}{z_1} \geq H_1'(z_1) + G_1'(z_1) = H_2'(z_1),$$

where the inequality follows from the concavity of G_1 and $G(0+) = 0$. Hence, H_2 is decreasing at $y = z_1$, and therefore, $z_2 \leq z_1$. Since F is increasing, it follows that $x_2 = F^{-1}(z_2) \leq F^{-1}(z_1) = x_1$.

6.1.3 (General n). Similarly, $H_n = (h_n/\varphi) \circ F^{-1}$ can be shown to be concave on $[0, 1]$ and $[1, +\infty)$; and $\lim_{y \to +\infty} H_n'(y) = 0$. There exists unique $z_n > 1$ such that

$$H_n(z_n)/z_n = H_n'(z_n).$$

The smallest nonnegative concave majorant W_n of H_n on $[0, +\infty)$ coincides with the straight-line $L_n(y) = yH_n'(z_n)$ on $[0, z_n]$, and with H_n on $[z_n, +\infty)$. If $x_n \triangleq F^{-1}(z_n)$, then

$$V_n(x) = \varphi(x) W_n(F(x)) = \begin{cases} e^{-(x_n - x)\sqrt{2\beta}} h_n(x_n), & x \leq x_n, \\ h_n(x), & x > x_n, \end{cases} \tag{32}$$

and $\sigma_n = \inf\{t \geq 0 : X(t) \geq x_n\}$ in (25).

The mapping $y \mapsto H_n(y)/y$ is increasing on $[1, z_n]$, and decreasing on $[z_n, +\infty)$; and $z_n > 1$ is its maximizer. We can show as above that $1 < z_n \leq z_1 = e^2$. These facts can be used to compute x_n numerically.

6.2 Geometric Brownian motion. Suppose that X is a geometric Brownian motion in $\mathcal{I} = (0, +\infty)$ with dynamics $dX(t) = X(t)[\beta dt + \sigma dB(t)]$, $t \geq 0$, where β and σ are positive constants. Let the reward function in (12) be $h(x) = (K - x)^+$, $x > 0$ for some constant $K > 0$.

The functions in (17) are unique (up to positive multipliers) increasing and decreasing solutions of the ordinary differential equation $(\sigma^2/2)x^2 u''(x) + \beta x u'(x) = \beta u(x)$ for $x > 0$, where the right-hand side is the infinitesimal generator of X applied to a smooth function u. We let $\psi(x) = x$ and $\varphi(x) = x^{-c}$, where $c \triangleq 2\beta/\sigma^2$; thus

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^{1+c}, \qquad x > 0.$$

Note that $F(0+) = 0$, $F(+\infty) = +\infty$; namely, both 0 and $+\infty$ are natural boundaries for X. One can also check that both ℓ_0 and ℓ_∞ of (20) are zero. Hence, all V_n's are finite, and $(\tau_1^{(n)}, \ldots, \tau_n^{(n)})$ of (25) is an optimal multiple-stopping strategy for every $n \geq 1$, thanks to Proposition 5.2 and Corollary 5.1.

6.2.1 (n=1). By Proposition 5.2, we have $V_1(x) = \varphi(x) W_1(F(x))$ for every $x > 0$, where W_1 is the smallest nonnegative concave majorant of

$$H_1(y) \triangleq (h/\varphi)(F^{-1}(y)) = \left(K y^{c/(1+c)} - y\right)^+, \quad y > 0.$$

It can be shown that $H_1(0) \triangleq H_1(0+) = 0$. The mapping H_1 is strictly concave on $[0, K^{1+c}]$, vanishes on $[K^{1+c}, +\infty)$, and has global maximum at $z_1 \triangleq [cK/(1+c)]^{1+c} \in (0, K^{1+c})$. Therefore, its smallest nonnegative concave majorant W_1 coincides with H_1 on $[0, z_1]$ and is equal to the constant $H_1(z_1)$ on $[z_1, +\infty)$; see Figure 2(b). If we define $x_1 \triangleq F^{-1}(z_1) = cK/(1+c)$, then

$$V_1(x) = \varphi(x) W_1(F(x)) = \begin{cases} K - x, & 0 < x \leq x_1, \\ (x_1/x)^{2r/\sigma^2}(K - x_1), & x > x_1. \end{cases}$$

Since $\boldsymbol{\Gamma}_1 = F^{-1}((0, z_1]) = (0, x_1]$, we have $\sigma_1 = \inf\{t \geq 0 : X(t) \leq x_1\}$.

6.2.2 (n=2). By Proposition 5.4, we have $V_2(x) = \varphi(x) W_2(F(x))$, where W_2 is the smallest nonnegative concave majorant of $H_2 = H_1 + G_1$, and $G_1 \triangleq (g_1/\varphi) \circ F^{-1}$. Since g_1 is nonnegative and β-excessive, the function G_1 is nonnegative and concave by Proposition 5.1; therefore, it is also nondecreasing. Because $G_1 \leq W_1$, we also have $G_1(+\infty) \leq W_1(z_1)$ and $G_1(0+) = 0$; see Figure 2(c).

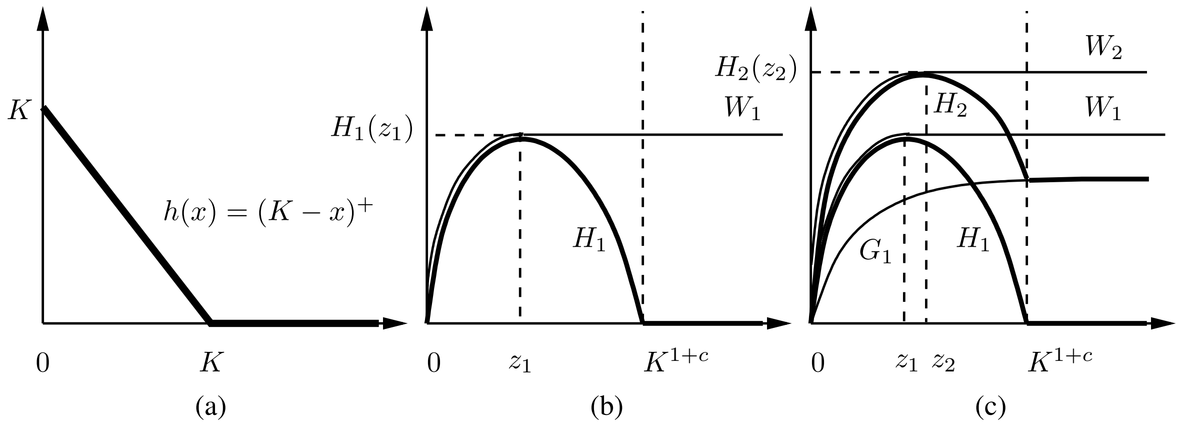

Figure 2: (Geometric Brownian motion). The sketches of (a) the reward function h, (b) the function H_1 and its smallest nonnegative concave majorant W_1, (c) $H_2 = H_1 + G_1$ and its smallest nonnegative concave majorant W_2.

Now observe that H_2 is the sum of two concave functions on $[0, K^{1+c}]$ and $[K^{1+c}, +\infty)$; therefore, it is itself concave on both intervals. We have $H_2(0+) = 0$. The function H_2 coincides with G_1 on $[K^{1+c}, +\infty)$ and has unique global maximum at some $z_2 \in (0, K^{1+c})$. Therefore, W_2 is the same as H_2 on $[0, z_2]$ and is equal to the constant $H_2(z_2)$ on $[z_2, +\infty)$. If $x_2 \triangleq F^{-1}(z_2)$, then $\sigma_2 = \inf\{t \geq 0 : X(t) \leq x_2\}$ and

$$V_2(x) = \varphi(x) W_2(F(x)) = \begin{cases} h_2(x), & 0 < x < x_2, \\ (x_2/x)^{2\beta/\sigma^2}\, h_2(x_2), & x \geq x_2. \end{cases}$$

Next let us show that $x_1 \leq x_2 < K$. Since z_n is unique global maximizer of H_n for $n = 1$ and $n = 2$, we have

$$0 \leq H_2(z_2) - H_2(z_1) = -(H_1(z_1) - H_1(z_2)) + (G_1(z_2) - G_1(z_1)),$$

which implies $G_1(z_2) - G_1(z_1) \geq H_1(z_1) - H_1(z_2) \geq 0$. Since G_1 is nondecreasing, we must have $z_1 \leq z_2 < K^{1+c}$. Because F is increasing, the inequalities $x_1 \leq x_2 < K$ follow.

One can check that the same results hold for general n. Namely, H_n is concave on $[0, K^{1+c}]$ and $[K^{1+c}, +\infty)$. It coincides on $[K^{1+c}, +\infty)$ with the bounded, nonnegative, nondecreasing, and concave function $G_{n-1} \triangleq (g_{n-1}/\varphi) \circ F^{-1}$, and we have $H_n(0+) = 0$. Therefore, H_n has a global maximum z_n, which is located in $(0, K^{1+c})$; in fact, $z_1 \leq z_n < K^{1+c}$. The smallest nonnegative concave majorant W_n of H_n coincides with H_n on $[0, z_n]$ and is equal to the constant $H_n(z_n)$ on $[z_n, +\infty)$. If we define $x_n \triangleq F^{-1}(z_n)$, then $\sigma_n = \inf\{t \geq 0 : X(t) \leq x_n\}$ is the nth stopping time in (25), and

$$V_n(x) = \varphi(x) W_n(F(x)) = \begin{cases} h_n(x), & 0 < x < x_n, \\ (x_n/x)^{2\beta/\sigma^2}\, h_n(x_n), & x \geq x_n. \end{cases}$$

6.3 Ornstein-Uhlenbeck process. Let X be the diffusion process in $\mathbb{R}$ with dynamics $dX_t = k(m - X_t)dt + \sigma dB_t$, $t \geq 0$, where $k > 0$, $\sigma > 0$, and $m \in \mathbb{R}$ are constants. Let the reward function in (12) be $h(x) = (e^x - L)^+$, $x \in \mathbb{R}$.

We shall denote by $\psi(\cdot)$ and $\varphi(\cdot)$ the functions in (17) for X, and by $\widetilde{\psi}(\cdot)$ and $\widetilde{\varphi}(\cdot)$ those for the process $Z_t \triangleq (X_t - m)/\sigma$, $t \geq 0$, which satisfies $dZ_t = -kZ_t + dB_t$, $t \geq 0$. For every $x \in \mathbb{R}$,

$$\widetilde{\psi}(x) = e^{kx^2/2} \mathcal{D}_{-\beta/k}(-x\sqrt{2k}) \quad \text{and} \quad \widetilde{\varphi}(x) = e^{kx^2/2} \mathcal{D}_{-\beta/k}(x\sqrt{2k}), \tag{33}$$

and $\psi(x) = \widetilde{\psi}((x-m)/\sigma)$ and $\varphi(x) = \widetilde{\varphi}((x-m)/\sigma)$, where $\mathcal{D}_\nu(\cdot)$ is the parabolic cylinder function; see Borodin and Salminen [6, Appendices 1.24 and 2.9]. The boundaries $\pm\infty$ are natural for X. By using the relation

$$\mathcal{D}_\nu(z) = 2^{-\nu/2} e^{-z^2/4} \mathcal{H}_\nu(z/\sqrt{2}), \quad z \in \mathbb{R} \tag{34}$$

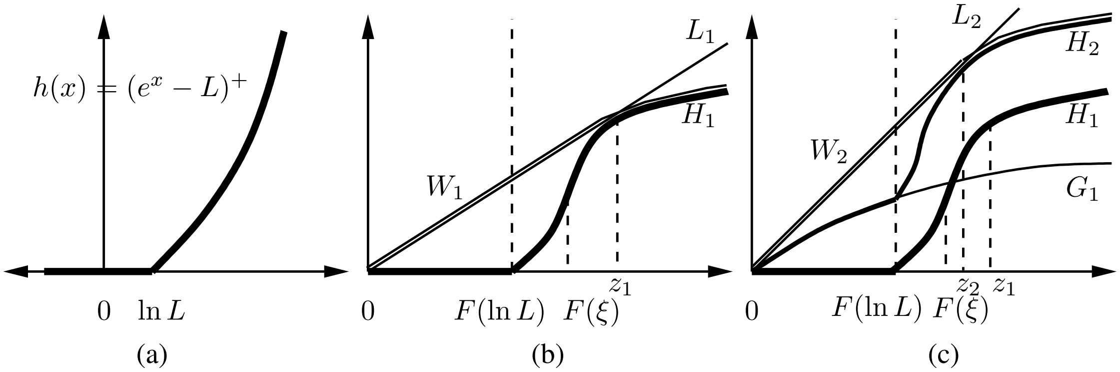

Figure 3: (Ornstein-Uhlenbeck process). The sketches of (a) the reward function h, (b) the function H_1 and its smallest nonnegative concave majorant W_1, (c) $H_2 = H_1 + G_1$ and its smallest nonnegative concave majorant W_2. In the figure, ξ is shown to be larger than $\ln L$.

in terms of Hermite function $\mathcal{H}_\nu(\cdot)$ of degree ν and its integral representation

$$\mathcal{H}_\nu(z) = \frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-t^2-2tz} t^{-\nu-1} dt, \quad \operatorname{Re}\nu < 0 \tag{35}$$

(see, for example, Lebedev [23, pp. 284, 290]), one can check that both limits in (20) are zero. By Proposition 5.2 and Corollary 5.1, the value function $V_n(\cdot)$ in (12) is finite, and the strategy $(\tau_1, \ldots, \tau_n)$ of (25) is optimal for every $n \geq 1$.

6.3.1 (n=1). This case, namely, pricing perpetual American call option on an asset with price process e^{X_t}, $t \geq 0$, has been recently studied by Cadenillas, Elliott, and Léger [7] by using variational inequalities. Let $F(x) \triangleq \psi(x)/\varphi(x)$ for every $x \in \mathbb{R}$. Since the reward function $h(\cdot)$ is increasing, the function $H_1(y) \triangleq (h/\varphi)(F^{-1}(y))$, $y \in (0, +\infty)$ is also increasing. Dayanik and Karatzas [13, Section 6] show that $H''(y)$ and $[(\mathcal{A} - \beta)h](F^{-1}(y))$ have the same sign at every y where h is twice-differentiable. Here, $(\mathcal{A} - \beta)h(x) = e^x[(\sigma^2/2) + km - \beta - kx] + \beta L$ for $x > \ln L$. Hence, there exists some $\xi > 0$ such that $H(\cdot)$ is convex on $[0, F(\xi \vee \ln L)]$ and concave on $[F(\xi \vee \ln L), +\infty)$; see Figure 3(b). It can also be checked that $H'(+\infty) = 0$ by using (34), (35) and the identity $\mathcal{H}'_\nu(z) = 2\nu\mathcal{H}_{\nu-1}(z)$, $z \in \mathbb{R}$; see Lebedev [23, p. 289], Borodin and Salminen [6, Appendix 2.9]. Therefore, there exists unique $z_1 > F(L)$ such that $H'(z_1) \triangleq H(z_1)/z_1$. The smallest nonnegative concave majorant $W_1(\cdot)$ of $H_1(\cdot)$ on $[0, \infty)$ coincides with the straight line $L_1(y) \triangleq (y/z_1)H_1(z_1)$, $y \geq 0$ on $[0, z_1]$, and with $H_1(\cdot)$ on $[z_1, +\infty)$. If $x_1 \triangleq F^{-1}(z_1)$, then the relation $V_1(x) = \varphi(x)W_1(F(x))$, $x \in \mathbb{R}$ gives

$$V_1(x) = \begin{cases} (e^{x_1} - L)\, e^{\frac{k}{2}\left[\left(\frac{x-m}{\sigma}\right)^2 - \left(\frac{x_1-m}{\sigma}\right)^2\right]} \dfrac{\mathcal{D}_{-\beta/k}\left(-\frac{x-m}{\sigma}\sqrt{2k}\right)}{\mathcal{D}_{-\beta/k}\left(-\frac{x_1-m}{\sigma}\sqrt{2k}\right)}, & x < x_1, \\ e^x - L, & x \geq x_1. \end{cases} \tag{36}$$

The stopping time $\sigma_1 = \inf\{t \geq 0 : X_t > x_1\}$ is the first exit time from $(0, x_1]$.

6.3.2 ($n \geq 2$). The analysis is similar to that in previous examples; compare, for example, Figures 4 and 3. The nth value function V_n in (12) is the same as the function in (36) except that x_1 is replaced with $x_n \triangleq F^{-1}(z_n)$ for every $n \geq 1$, and $\sigma_n = \inf\{t \geq 0 : X_t > x_n\}$ in (25), where z_n is the unique solution of $H'_n(y) = H_n(y)/y$, $y \geq 0$. The critical value z_n is the unique maximum of $y \mapsto H_n(y)/y$ and is contained in $(F(\ln L), z_1)$. It can be calculated numerically.

6.4 Another mean reverting diffusion. Let X be a diffusion process in $(0, +\infty)$ with dynamics

$$dX_t = \mu X_t(\alpha - X_t)dt + \sigma X_t dB_t, \quad t \geq 0, \tag{37}$$

and $h(x) \triangleq (x - K)^+$ for every $x > 0$ in (12), where μ, α, σ and K are positive constants. The process has been studied widely in irreversible investment and harvesting problems; see, for example, Dixit and Pindyck [14], Alvarez and Shepp [2].

The functions $\psi(\cdot)$ and $\varphi(\cdot)$ in (17) are the increasing and decreasing fundamental solutions of $(1/2)\sigma^2 x^2 u''(x) + \mu x(\alpha - x)u'(x) - \beta u(x) = 0$, respectively. Denote by

$$M(a,b,x) \triangleq \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^n}{n!}, \quad (a)_k \triangleq a(a+1)\cdots(a+k-1),\ (a)_0 = 1, \tag{38}$$

$$U(a,b,x) \triangleq \frac{\pi}{\sin \pi b}\left\{\frac{M(a,b,x)}{\Gamma(1+a-b)\Gamma(b)} - x^{1-b}\frac{M(1+a-b,2-b,x)}{\Gamma(a)\Gamma(2-b)}\right\} \tag{39}$$

the confluent hypergeometric functions of the first and second kind, respectively, which are two linearly independent solutions for the Kummer equation $xw''(x) + (b-x)w'(x) - ax = 0$ for arbitrary positive constants a and b; see, for example, Abramowitz and Stegun [1, Chapter 13]. Then

$$\psi(x) \triangleq (cx)^{\theta^+} M(\theta^+, a^+, cx), \quad \text{and} \quad \varphi(x) \triangleq (cx)^{\theta^+} U(\theta^+, a^+, cx), \qquad x > 0,$$

and

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = \frac{M(\theta^+, a^+, cx)}{U(\theta^+, a^+, cx)}, \quad x > 0,$$

where $c \triangleq 2\mu\sigma^2$, $a^{\pm} = 2\theta^{\pm} + (2\mu\alpha/\sigma^2)$, and

$$\theta^{\pm} \triangleq \left(\frac{1}{2} - \frac{\mu\alpha}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{\mu\alpha}{\sigma^2}\right) + \frac{2\beta}{\sigma^2}}, \quad \theta^- < 0 < \theta^+$$

are the roots of the equation $(1/2)\sigma^2\theta(\theta-1) + \mu\alpha\theta - \beta = 0$; see Dayanik and Karatzas [13]. Since $\psi(+\infty) = \varphi(-\infty) = +\infty$, the boundaries 0 and $+\infty$ are natural. Both limits in (20) are zero. By Proposition 5.2, all the V_n's are finite, and $(\tau_1^{(n)}, \ldots, \tau_n^{(n)})$ of (25) is optimal for every $n \geq 1$ because of Corollary 5.1.

6.4.1 (n=1). Dayanik and Karatzas [13, Section 6.10] show that the function $H_1 = (h/\varphi) \circ F^{-1}$ is increasing, convex on $[0, F(K \vee \xi)]$, and concave on $[F(K \vee \xi), +\infty)$ for some $\xi > 0$, and $H'(+\infty) = 0$; see Figure 4. Therefore, $H(y)/y = H'(y)$ has unique solution—call it z_1, and the smallest nonnegative concave majorant W_1 of H_1 coincides with the straight-line $L_1(y) = (y/z_1)H(z_1)$ on $[0, z_1]$, and with H_1 on $[z_1, +\infty)$. If we set $x_1 \triangleq F^{-1}(z_1) > K$, then

$$V_1(x) = \varphi(x)W_1(F(x)) = \begin{cases} \left(\dfrac{x}{x_1}\right)^{\theta^+} \dfrac{M(\theta^+, a^+, cx)}{M(\theta^+, a^+, cx_1)}(x_1 - K), & 0 < x < x_1, \\ x - K, & x > x_1. \end{cases} \tag{40}$$

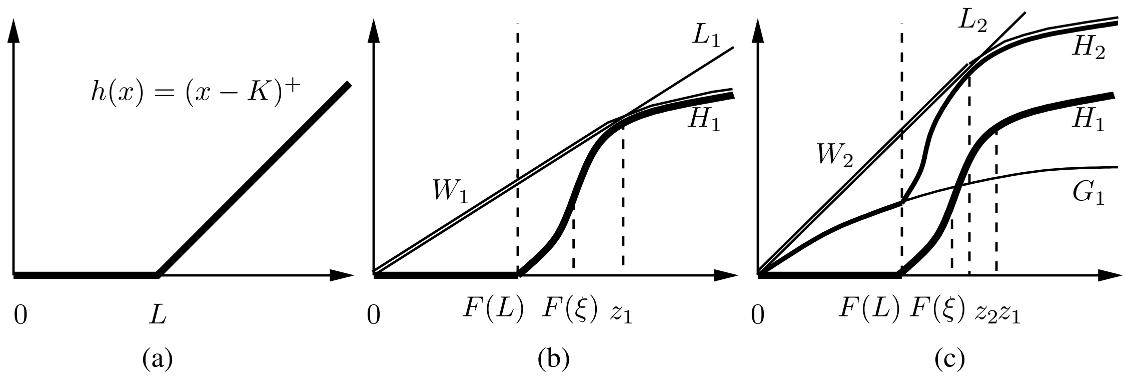

Figure 4: (Mean-reverting process). The sketches of (a) the reward function h, (b) the function H_1 and its smallest nonnegative concave majorant W_1, (c) $H_2 = H_1 + G_1$ and its smallest nonnegative concave majorant W_2. In the figure, ξ is shown to be larger than K.

6.4.2 ($n \geq 2$). The fundamental properties of the functions W_1 and H_1 are essentially the same as those in the first example; compare the graphs in Figures 4 and 1. Therefore, the analysis is the same as that in Section 6.1.2 and 6.1.3 after obvious changes, such as, instead of (30) and (32), we have

$$V_n(x) = \varphi(x) W_n(F(x)) = \begin{cases} \left(\dfrac{x}{x_n}\right)^{\theta^+} \dfrac{M(\theta^+, a^+, cx)}{M(\theta^+, a^+, cx_n)}\, h_n(x_n), & 0 < x < x_n, \\ h_n(x), & x > x_n \end{cases} \tag{41}$$

for $n \geq 2$. Finally, the nth stopping time $\sigma_n = \inf\{t \geq 0 : X_t \geq x_n\}$ in (25) is the first hitting time of X to $[x_n, +\infty)$. Moreover, $x_n = F^{-1}(z_n)$; the number z_n is the unique maximum of $y \mapsto H_n(y)/y$ and is contained in (K, x_1). Therefore, z_n and x_n can be calculated numerically.

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