THE STANDARD POISSON DISORDER PROBLEM REVISITED

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Abstract. A change in the arrival rate of a Poisson process sometimes necessitates immediate action. If the change time is unobservable, then the design of online change detection procedures becomes important and is known as the Poisson disorder problem. Formulated and partially solved by Davis [Banach Center Publ., 1:65–72, 1976], the *standard Poisson problem* addresses the tradeoff between false alarms and detection delay costs in the most useful way for applications. In this paper we solve the standard problem completely and describe efficient numerical methods to calculate the policy parameters.

1. Introduction

Suppose that the rate of a Poisson process X changes from one known value to another at a random and unobservable time θ, which is nonnegative and has exponential distribution $\mathbb{P}\{\theta > t\} = (1-\pi)e^{-\lambda t}$, $t \geq 0$. The classical Poisson disorder problem is to detect the *disorder time* θ as quickly as possible. The detection rule is typically a stopping time τ of the history generated by the process X, and minimizes a suitable measure of the expected losses due to false alarms on the event $\{\tau < \theta\}$ and the detection delay $(\tau - \theta)^+$, e.g.,

$$
\begin{aligned}
&R^{(1)}_\tau(\pi) \triangleq \mathbb{P}\{\tau < \theta - \varepsilon\} + c\,\mathbb{E}(\tau-\theta)^+, \quad && R^{(2)}_\tau(\pi) \triangleq \mathbb{P}\{\tau < \theta\} + c\,\mathbb{E}(\tau-\theta)^+, \\
&R^{(3)}_\tau(\pi) \triangleq \mathbb{E}(\theta-\tau)^+ + c\,\mathbb{E}(\tau-\theta)^+, && R^{(4)}_\tau(\pi) \triangleq \mathbb{P}\{\tau < \theta\} + c\,\mathbb{E}[e^{\alpha(\tau-\theta)^+} - 1],
\end{aligned} \tag{1.1}
$$

for some positive constants ε, α and c. The first three criteria model the *detection delay* cost by a linear function of the delay time and are suitable, e.g., for capturing the cost of defective merchandise produced by an undetected out-of-control industrial process. The fourth criterion penalizes the delay time exponentially; especially in financial applications, it gives a better account for the unrealized revenues due to the lost investment opportunities over the delay time. The *false alarms* are also weighted differently; the third criterion

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minimizes the expected total miss, while the other criteria incorporate the *frequency* of false alarms (outside the acceptable window $(\theta - \varepsilon, \theta]$ in the case of the first criterion).

All of the criteria in (1.1) are in fact special instances of the so-called *standard Poisson disorder problem*; namely, they can be cast in the form

$$(1.2) \quad \mathfrak{R}_\tau(\pi; \Phi, k) \triangleq \gamma(\pi) + \beta(\pi)\, \mathbb{E}_0 \int_0^\tau e^{-\lambda t}(\Phi_t - k)dt \quad \text{for every stopping time } \tau \text{ of } X,$$

for some known constant $k > 0$, known functions γ, β from $[0,1)$ into $\mathbb{R}_+$, and some suitable process $\Phi = \{\Phi_t : t \geq 0\}$ which is adapted to the history of X and plays the rôle of appropriate "odds-ratio" process. We have denoted by $\mathbb{P}_0$ a probability measure which is equivalent to $\mathbb{P}$ on each finite time-interval $[0, t]$, and under which the observed process X becomes a Poisson process with rate λ_0; see (2.4) for a detailed description. Finally, $\mathbb{E}_0$ denotes expectation with respect to $\mathbb{P}_0$.

Under the original probability measure $\mathbb{P}$ and in a form similar to (1.2), the resemblance of the criteria $R^{(1)}$ and $R^{(3)}$ (also, $R^{(2)}$ as a special case of $R^{(1)}$ with $\varepsilon = 0$) was first noticed by Davis (1976), who also coined the term "standard" for the Poisson disorder problems with a criterion admitting his general representation. Using the theory of filtering for point processes, Davis (1976) partially solved the standard Poisson disorder problem and improved the partial solution of Galchuk and Rozovskii (1971) for the criterion $R^{(2)}$ in (1.1).

In this paper we provide the *complete* solution of the standard Poisson disorder problem. The process Φ in (1.2) turns out to be a piecewise-deterministic Markov process (see, e.g., Davis (1993; 1984)). Thus, the minimization of (1.2) over all stopping times τ of the process X becomes a discounted optimal stopping problem for the Markov process Φ. We formulate and solve a related differential-delay equation with a free boundary: the optimal detection rule is to set off the alarm as soon as the process Φ reaches or exceeds a suitable threshold. We also describe a straightforward and accurate numerical procedure to calculate the critical threshold and the minimum cost function.

Two special cases $R^{(2)}$ and $R^{(4)}$ in (1.2) have been recently solved by Peskir and Shiryaev (2002) and Bayraktar and Dayanik (2003), respectively. Peskir and Shiryaev (2002) work with the posterior probability process $\Pi_t \triangleq \mathbb{P}\{\theta \leq t | X_s, 0 \leq s \leq t\}$, $t \geq 0$ (instead of the odds-ratio process $\Phi_t \triangleq \Pi_t/(1 - \Pi_t)$). This increases considerably the mathematical difficulty, preventing their analysis from reaching its full capacity. Working with the odds-ratio process Φ instead, Bayraktar and Dayanik (2003) were able to reveal the complete structure of the solution for the (apparently) more difficult problem with exponential delay cost: under the original probability measure $\mathbb{P}$, the detection problem with $R^{(4)}$ reduces to an optimal stopping problem for a *two-dimensional* piecewise-deterministic Markov process. Here, we also show the true one-dimensional nature of that problem by the new formulation under the auxiliary probability measure $\mathbb{P}_0$ under which we have taken the expectation in (1.2).

In the next section we give a precise description of the model, and formulate an equivalent optimal stopping problem. In Section 3, we solve the optimal stopping problem and describe a numerical method to calculate the policy parameters.

2. The problem description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a counting process $X = \{X_t, t \geq 0\}$ and a random variable θ with the distribution

$$\mathbb{P}\{\theta = 0\} = \pi \quad \text{and} \quad \mathbb{P}\{\theta > t\} = (1-\pi)e^{-\lambda t}, \quad 0 \leq t < \infty \tag{2.1}$$

for some known constants $\pi \in [0,1)$, $\lambda > 0$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ of X, enlarged by $\mathbb{P}$-null sets so as to satisfy the usual conditions, and consider the larger filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ with $\mathcal{G}_t \triangleq \mathcal{F}_t \vee \sigma(\theta)$. If θ is known, the process X is a Poisson process with rate λ_0 on the time interval $[0, \theta]$ and with rate λ_1 on (θ, ∞) for some known positive constants λ_0 and λ_1. Namely, the process X is a counting process such that

$$X_t - \int_0^t \left[\lambda_0 1_{\{s<\theta\}} + \lambda_1 1_{\{s\geq\theta\}}\right] ds, \; t \geq 0 \qquad \text{is a } (\mathbb{P}, \mathbb{G})\text{-martingale;} \tag{2.2}$$

see, for instance, Brémaud (1981; 1975), Brémaud and Jacod (1977). The crucial feature here, is that θ is neither known nor observable; only the process X is observable. Our problem is to find a quickest detection rule for the disorder time θ, which is *adapted to* the history $\mathbb{F}$ generated by the observed process X. If such a rule exists, then it is typically an $\mathbb{F}$-stopping time minimizing a suitable error criterion. Before we specify this criterion, we shall first describe a useful reference probability measure $\mathbb{P}_0$ as follows.

The Model. Let us start with a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$ which supports a Poisson process X with rate λ_0 and an *independent* random variable θ with the distribution $\mathbb{P}_0\{\theta = 0\} = \pi$ and $\mathbb{P}_0\{\theta > 0\} = (1-\pi)e^{-\lambda t}$, $t > 0$. Let the natural filtration $\mathbb{F}$ of X and its augmentation $\mathbb{G}$ by $\sigma(\theta)$ be defined as above. Expressed in terms of the right-continuous, $\mathbb{G}$-adapted process $h(t) \triangleq \lambda_0 1_{\{t<\theta\}} + \lambda_1 1_{\{t\geq\theta\}}$, $0 \leq t < \infty$ (the integrand of (2.2)), the

$(\mathbb{P}_0, \mathbb{G})$-martingale

$$Z_t \triangleq \exp\left\{ \int_0^t \log\left(\frac{h(s)}{\lambda_0}\right) dX_s - \int_0^t [h(s) - \lambda_0] ds \right\}, \quad t \geq 0 \tag{2.3}$$

induces a new probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ which satisfies

$$\left.\frac{d\mathbb{P}}{d\mathbb{P}_0}\right|_{\mathcal{G}_t} = Z_t = 1_{\{\theta > t\}} + 1_{\{\theta \leq t\}} \frac{L_t}{L_\theta} \tag{2.4}$$

for every $0 \leq t < \infty$, where

$$L_t \triangleq \left(\frac{\lambda_1}{\lambda_0}\right)^{X_t} e^{-(\lambda_1 - \lambda_0)t}. \tag{2.5}$$

We take $\mathcal{F} = \sigma\left(\cup_{t \geq 0} \mathcal{G}_t\right)$, without loss of generality. Under the new probability measure $\mathbb{P}$, the process X has the ($\mathbb{G}$-progressively measurable) intensity $h(\cdot)$. This is to say that (2.2) holds; see, e.g., Brémaud (1981; 1972), Brémaud and Jacod (1977). Since $\mathbb{P}$ and $\mathbb{P}_0$ coincide on $\mathcal{G}_0 = \sigma(\theta)$, we conclude that (2.1) also holds. Therefore, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the random elements X and θ have the same properties posited at the beginning of this section, and we shall assume henceforth that they are as described here.

We shall denote by $\mathcal{S}$ the collection of all $\mathbb{F}$-stopping times. Let us also introduce the posterior probability $\Pi_t \triangleq \mathbb{P}\{\theta \leq t | \mathcal{F}_t\}$, $t \geq 0$ that the disorder has happened at or before time t, given all past observations of X, and the *generalized odds-ratio processes*

$$\Phi_t^{(\alpha)} \triangleq \frac{\mathbb{E}[e^{\alpha(t-\theta)} 1_{\{\theta \leq t\}} | \mathcal{F}_t]}{1 - \Pi_t}, \qquad 0 \leq t < \infty \tag{2.6}$$

for $\alpha \in [0, \infty)$. The standard Poisson disorder problem is then to calculate the *minimum Bayes risk*

$$V(\pi; \Phi^{(\alpha)}, k) \triangleq \inf_{\tau \in \mathcal{S}} \mathfrak{R}_\tau(\pi; \Phi^{(\alpha)}, k), \quad \pi \in [0, 1) \tag{2.7}$$

with $\mathfrak{R}$ as in (1.2), and to find a stopping time $\tau \in \mathcal{S}$ which attains the infimum in (2.7). If such a stopping time exists, it is called an *optimal Bayes detection rule.*

Proposition 2.1. *For every $\pi \in [0, 1)$ and $\tau \in \mathcal{S}$, we have $R_\tau^{(i)}(\pi) = \mathfrak{R}_\tau(\pi; \Phi^{(0)}, k_i)$, $i = 1, 2, 3$, and $R_\tau^{(4)}(\pi) = \mathfrak{R}_\tau(\pi; \Phi^{(\alpha)}, k_4)$ for every positive α, where $k_1 = (\lambda/c)e^{-\varepsilon\lambda}$, $k_2 = \lambda/c$,*

$k_3 = 1/c$, $k_4 = \lambda/(c\alpha)$. *More precisely, we have*

$$\begin{aligned} R^{(1)}_\tau(\pi) &= (1-\pi)e^{-\lambda\varepsilon} + c(1-\pi)\,\mathbb{E}_0 \int_0^\tau e^{-\lambda t}\left[\Phi^{(0)}_t - (\lambda/c)e^{-\lambda\varepsilon}\right]dt, \\ R^{(3)}_\tau(\pi) &= (1-\pi)/\lambda + c(1-\pi)\,\mathbb{E}_0 \int_0^\tau e^{-\lambda t}\left[\Phi^{(0)}_t - (1/c)\right]dt, \\ R^{(4)}_\tau(\pi) &= (1-\pi) + c\alpha(1-\pi)\,\mathbb{E}_0 \int_0^\tau e^{-\lambda t}\left[\Phi^{(\alpha)}_t - (\lambda/(c\alpha))\right]dt, \quad \alpha > 0, \end{aligned} \tag{2.8}$$

and $R^{(2)}$ *is the same as* $R^{(1)}$ *with* $\varepsilon = 0$.

Before supplying the proof, let us first derive the dynamics of the processes $\Phi^{(\alpha)}$, $\alpha \geq 0$. By the Bayes rule (see, e.g., Lipster and Shiryaev (2001, Section 7.9)) and the independence of θ and $\mathbb{F}$ under $\mathbb{P}_0$, we have

$$\Pi_t = \mathbb{P}\{\theta \leq t | \mathcal{F}_t\} = \frac{\mathbb{E}_0[Z_t 1_{\{\theta\leq t\}}|\mathcal{F}_t]}{\mathbb{E}_0[Z_t|\mathcal{F}_t]} \quad \text{and} \quad 1-\Pi_t = \frac{\mathbb{E}_0[1_{\{\theta>t\}}|\mathcal{F}_t]}{\mathbb{E}_0[Z_t|\mathcal{F}_t]} = \frac{(1-\pi)e^{-\lambda t}}{\mathbb{E}_0[Z_t|\mathcal{F}_t]} \tag{2.9}$$

for every $t \geq 0$. From (2.6), (2.9) and (2.4), it follows

$$\begin{aligned} \Phi^{(\alpha)}_t &= \frac{\mathbb{E}[e^{\alpha(t-\theta)}1_{\{\theta\leq t\}}|\mathcal{F}_t]}{1-\Pi_t} = \frac{\mathbb{E}_0[Z_t e^{\alpha(t-\theta)}1_{\{\theta\leq t\}}|\mathcal{F}_t]}{(1-\Pi_t)\mathbb{E}_0[Z_t|\mathcal{F}_t]} = \frac{e^{\lambda t}}{1-\pi}\,\mathbb{E}_0[Z_t e^{\alpha(t-\theta)}1_{\{\theta\leq t\}}|\mathcal{F}_t] \\ &= \frac{e^{(\lambda+\alpha)t}}{1-\pi}\left[\pi L_t + (1-\pi)\int_0^t \frac{L_t}{L_s}\lambda e^{-(\lambda+\alpha)s}ds\right] = U^{(\alpha)}_t + V^{(\alpha)}_t, \end{aligned} \tag{2.10}$$

where

$$U^{(\alpha)}_t \triangleq \frac{\pi}{1-\pi}e^{(\lambda+\alpha)t}L_t \qquad \text{and} \qquad V^{(\alpha)}_t \triangleq e^{(\lambda+\alpha)t}L_t \int_0^t \frac{1}{L_s}\lambda e^{-(\lambda+\alpha)s}ds$$

for every $t \geq 0$. The process $L = \{L_t, t \geq 0\}$ in (2.4) is a $(\mathbb{P}_0, \mathbb{F})$-martingale and is the unique locally bounded solution of the equation

$$dL_t = [(\lambda_1/\lambda_0) - 1]L_{t-}(dX_t - \lambda_0 dt), \qquad L_0 = 1;$$

see, e.g., Jacod and Shiryaev (2003, Theorem 4.61, p. 59) and Revuz and Yor (1999, Proposition 4.7, p. 6). By means of the chain-rule, we obtain

$$\begin{aligned} dU^{(\alpha)}_t &= (\lambda+\alpha-\lambda_1+\lambda_0)U^{(\alpha)}_t dt + [(\lambda_1/\lambda_0)-1]U^{(\alpha)}_{t-}dX_t, & U^{(\alpha)}_0 &= \pi/(1-\pi), \\ dV^{(\alpha)}_t &= \left(\lambda + (\lambda+\alpha-\lambda_1+\lambda_0)V^{(\alpha)}_t\right)dt + [(\lambda_1/\lambda_0)-1]V^{(\alpha)}_{t-}dX_t, & V^{(\alpha)}_0 &= 0. \end{aligned}$$

Therefore, for every $\alpha \geq 0$, the process $\Phi^{(\alpha)}_t = U^{(\alpha)}_t + V^{(\alpha)}_t$, $t \geq 0$ satisfies

$$d\Phi^{(\alpha)}_t = \left(\lambda + (\lambda+\alpha-\lambda_1+\lambda_0)\Phi^{(\alpha)}_t\right)dt + [(\lambda_1/\lambda_0)-1]\Phi^{(\alpha)}_{t-}dX_t, \quad \Phi^{(\alpha)}_0 = \pi/(1-\pi). \tag{2.11}$$

Proof of Proposition 2.1. By setting $\alpha = 0$ in (2.10), we obtain $\mathbb{E}_0[Z_t 1_{\{\theta \le t\}} | \mathcal{F}_t] = (1 - \pi) e^{-\lambda t} \Phi_t^{(0)}$, $t \ge 0$. Therefore,

$$\begin{aligned} \mathbb{E}[(\tau-\theta)^+] &= \mathbb{E}\left[1_{\{\tau>\theta\}} \int_\theta^\tau dt\right] = \mathbb{E}\int_0^\infty 1_{\{\tau>t\}} 1_{\{\theta \le t\}} dt = \int_0^\infty \mathbb{E}_0[Z_t 1_{\{\tau>t\}} 1_{\{\theta\le t\}}] dt \\ &= \int_0^\infty \mathbb{E}_0[1_{\{\tau>t\}} \mathbb{E}_0[Z_t 1_{\{\theta\le t\}} | \mathcal{F}_t]] dt = (1-\pi)\, \mathbb{E}_0 \int_0^\tau e^{-\lambda t} \Phi_t^{(0)} dt, \end{aligned} \tag{2.12}$$

for every $\tau \in \mathcal{S}$. On the other hand, for every $\varepsilon \ge 0$ and every $\mathbb{F}$-stopping time τ which takes countably many values, say in $\{t_n, n \in \mathbb{N}\}$ for distinct $t_n \in \mathbb{R}_+ \cup \{+\infty\}$, we have

$$\begin{aligned} \mathbb{P}\{\tau < \theta - \varepsilon\} &= \sum_n \mathbb{P}\{t_n < \theta - \varepsilon, \tau = t_n\} = \sum_n \mathbb{E}_0\left[Z_{t_n} 1_{\{t_n < \theta - \varepsilon\}} 1_{\{\tau = t_n\}}\right] \\ &= \sum_n \mathbb{E}_0\left[1_{\{\theta > t_n + \varepsilon\}} 1_{\{\tau = t_n\}}\right] = \sum_n (1-\pi) e^{-(\lambda+\varepsilon) t_n}\, \mathbb{E}_0[1_{\{\tau = t_n\}}] \\ &= (1-\pi) e^{-\lambda\varepsilon}\, \mathbb{E}_0 \sum_n e^{-\lambda t_n} 1_{\{\tau = t_n\}} = (1-\pi) e^{-\lambda\varepsilon}\, \mathbb{E}_0 \sum_n \left[1 - \int_0^{t_n} \lambda e^{-\lambda t} dt\right] 1_{\{\tau = t_n\}} \\ &= (1-\pi) e^{-\lambda\varepsilon} - (1-\pi)\lambda e^{-\lambda\varepsilon}\, \mathbb{E}_0 \int_0^\tau e^{-\lambda t} dt. \end{aligned} \tag{2.13}$$

The third equation follows from the expression for Z_t in (2.4), and the fourth from the independence of θ and $\mathcal{F}_\infty$ under $\mathbb{P}_0$.

Now an arbitrary $\mathbb{F}$-stopping time τ is the almost-sure limit of a decreasing sequence $\{\tau_n\}_{n\ge1}$ of $\mathbb{F}$-stopping times which take countably many values, and the above equality (2.13) holds for every τ_n. Since $t \mapsto 1_{\{t<\theta-\varepsilon\}}$ and $t \mapsto \int_0^t e^{-\lambda s} ds$ are bounded and right-continuous, (2.13) also holds for τ, because of the bounded convergence theorem, after passing to the limit on both sides.

Multiplying (2.12) by c and summing that with (2.13), we obtain $R^{(1)}$ in (2.8), with $\gamma(\pi) = (1-\pi)e^{-\lambda\varepsilon}$, $\beta(\pi) = c(1-\pi)$ and $k_1 = (\lambda/c)e^{-\lambda\varepsilon}$ in (1.2). Similarly, $k_2 = \lambda/c$ if we set $\varepsilon = 0$ in $R^{(1)}$ to get $R^{(2)}$ of (1.1). On the other hand, for every $\mathbb{F}$-stopping time τ

$$\begin{aligned} \mathbb{E}[(\theta-\tau)^+] &= \mathbb{E}\left[1_{\{\tau<\theta\}} \int_\tau^\theta dt\right] = \mathbb{E}\int_0^\infty 1_{\{\theta>t\}} 1_{\{\tau\le t\}} dt = \int_0^\infty \mathbb{E}_0\left[Z_t 1_{\{\theta>t\}} 1_{\{\tau\le t\}}\right] dt \\ &= \int_0^\infty \mathbb{E}_0\left[1_{\{\theta>t\}} 1_{\{\tau\le t\}}\right] dt = (1-\pi)\int_0^\infty e^{-\lambda t}\left(1 - \mathbb{E}_0 1_{\{\tau>t\}}\right) dt \\ &= (1-\pi)\left[\frac{1}{\lambda} - \mathbb{E}_0 \int_0^\tau e^{-\lambda t} dt\right], \end{aligned} \tag{2.14}$$

where we have again used the independence of θ and $\mathcal{F}_\infty$ under $\mathbb{P}_0$. Note from (2.10) that $\mathbb{E}_0[Z_t e^{\alpha(t-\theta)} 1_{\{\theta\le t\}}|\mathcal{F}_t] = (1-\pi)e^{-\lambda t}\Phi_t^{(\alpha)}$, $t \ge 0$ and

$$\begin{aligned}(2.15)\quad \mathbb{E}[e^{\alpha(\tau-\theta)^+}-1] &= \alpha\,\mathbb{E}\left[1_{\{\tau>\theta\}}\int_\theta^\tau e^{\alpha(t-\theta)}dt\right] = \alpha\,\mathbb{E}\int_0^\infty 1_{\{\tau>t\}}1_{\{\theta\le t\}}e^{\alpha(t-\theta)}dt\\ &= \alpha\int_0^\infty \mathbb{E}_0[1_{\{\tau>t\}}Z_t e^{\alpha(t-\theta)}1_{\{\theta\le t\}}]dt = \alpha\int_0^\infty \mathbb{E}_0\left[1_{\{\tau>t\}}\mathbb{E}_0[Z_t 1_{\{\theta\le t\}}e^{\alpha(t-\theta)}|\mathcal{F}_t]\right]dt\\ &= \alpha(1-\pi)\,\mathbb{E}_0\int_0^\tau e^{-\lambda t}\Phi_t^{(\alpha)}dt, \quad \tau\in\mathcal{S},\ \alpha>0.\end{aligned}$$

From (2.12) with $\varepsilon = 0$, (2.14) and (2.15), we get $R^{(3)}$ and $R^{(4)}$ as in (2.8), with $k_3 = 1/c$ and $k_4 = \lambda/(c\alpha)$. □

It is clear from (2.11) that the process $\Phi^{(\alpha)} = \{\Phi_t^{(\alpha)}, t\ge 0\}$ is a (piecewise-deterministic) Markov process. For every bounded, continuous and continuously differentiable function $f : \mathbb{R}_+ \mapsto \mathbb{R}$, we have

$$\begin{aligned} f(\Phi_t^{(\alpha)}) - f(\Phi_0^{(\alpha)}) &= \sum_{0<s\le t}\left[f\big(\Phi_s^{(\alpha)}\big) - f\big(\Phi_{s-}^{(\alpha)}\big)\right] + \int_0^t f'\big(\Phi_s^{(\alpha)}\big)\left[\lambda + (\lambda+\alpha-\lambda_1+\lambda_0)\Phi_s^{(\alpha)}\right]ds\\ (2.16)\qquad &= \int_0^t \left[f\big((\lambda_1/\lambda_0)\Phi_{s-}^{(\alpha)}\big) - f\big(\Phi_{s-}^{(\alpha)}\big)\right](dX_s - \lambda_0 ds) + \int_0^t \mathcal{A}^{(\alpha)} f(\Phi_s^{(\alpha)})\,ds\end{aligned}$$

where

$$(2.17)\qquad \mathcal{A}^{(\alpha)} f(\phi) = [\lambda + (\lambda+\alpha-\lambda_1+\lambda_0)\phi]f'(\phi) + \lambda_0\left[f\big((\lambda_1/\lambda_0)\phi\big) - f(\phi)\right], \quad \phi > 0.$$

Since $\{X_t - \lambda_0 t,\ t\ge 0\}$ is a $(\mathbb{P}_0, \mathbb{F})$-martingale, we obtain from (2.16) that

$$\mathbb{E}_0 f(\Phi_t^{(\alpha)}) = f(\Phi_0^{(\alpha)}) + \mathbb{E}_0\int_0^t \mathcal{A}^{(\alpha)}(\Phi_s^{(\alpha)})ds, \qquad t\ge 0,$$

i.e., $\mathcal{A}^{(\alpha)}$ in (2.17) is the infinitesimal generator of X under $\mathbb{P}_0$, acting on bounded functions $f(\cdot)$ in $\mathcal{C}^1(\mathbb{R}_+)$. Thus, *the standard Poisson disorder problem (2.7), (1.2) has been cast as an optimal stopping problem for the Markov process $\Phi^{(\alpha)}$*. To solve it, we shall formulate in the next section a related differential-delay equation involving $\mathcal{A}^{(\alpha)}$ in (2.17) with a free boundary.

3. A free boundary problem and its solution

The problem of (2.6), (1.2) admits a very simple solution for a certain range of parameters, because of the special properties of the sample-paths of $\Phi^{(\alpha)}$. This was first noticed by

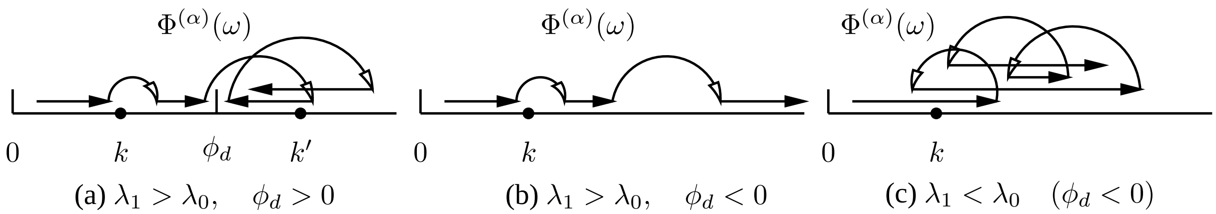

FIGURE 1. The behavior of the paths of the process $\Phi^{(\alpha)}$. The process $\Phi^{(\alpha)}$ jumps upwards (resp., downwards) if $\lambda_1 > \lambda_0$ (resp., $\lambda_1 < \lambda_0$). Between jumps, it always drifts away from the origin if $\phi_d < 0$, and reverts to ϕ_d if $\phi_d > 0$.

Davis (1976). We recall this solution here, for the sake of completeness. For all future references, let us record the basic notation:

$$a \triangleq \lambda + \alpha - \lambda_1 + \lambda_0, \quad b \triangleq \lambda + \lambda_0 > 0, \quad r \triangleq \lambda_1/\lambda_0, \quad \phi_d \triangleq -\lambda/a \neq 0. \tag{3.1}$$

Proposition 3.1 (**Case I**). *Suppose that $\lambda_1 \geq \lambda_0$, and either $\phi_d < 0$ or $0 < k \leq \phi_d$. Then the stopping rule*

$$\tau_k \triangleq \inf\{t \geq 0 : \Phi_t^{(\alpha)} \geq k\} \tag{3.2}$$

is optimal for (2.7).

Let $\sigma_0 \equiv 0$ and $\sigma_n \triangleq \inf\{t > \sigma_{n-1} : X_t - X_{t-} > 0\}$ be the n-th jump time of X for every $n \in \mathbb{N}$ (by convention, $\inf \varnothing = +\infty$). From (2.11), it is easy to obtain

$$\begin{aligned} &\Phi_t^{(\alpha)} = \phi_d + [\Phi_{\sigma_{n-1}}^{(\alpha)} - \phi_d] \exp\{-(\lambda/\phi_d)(t - \sigma_{n-1})\}, && \sigma_{n-1} \leq t < \sigma_n, \\ &\Phi_0^{(\alpha)} \in \mathbb{R}_+ \quad \text{and} \quad \Phi_{\sigma_n}^{(\alpha)} = r\Phi_{\sigma_n -}^{(\alpha)}, && n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

If $\phi_d < 0$, then the paths of the process $\Phi^{(\alpha)}$ always increase between jumps; see Figure 1(b,c).

If $\phi_d > 0$, then ϕ_d is the mean-level to which the process $\Phi^{(\alpha)}$ reverts between jumps; see Figure 1(a). The difference $\Phi_t^{(\alpha)} - \phi_d$ in (3.3) never vanishes before a jump, and $\Phi_{\sigma_n}^{(\alpha)} \neq \phi_d$ for all $n > 0$ almost surely. Moreover, $\Phi^{(\alpha)}$ has positive (respectively, negative) jumps if $\lambda_1 > \lambda_0$ (respectively, $\lambda_1 < \lambda_0$).

Under the hypotheses of Proposition 3.1, if $\Phi^{(\alpha)}$ leaves the interval $[0, k]$, then it does not return there; see Figure 1(a,b). Therefore, the form of $\mathfrak{R}_\tau(\pi; \Phi^{(\alpha)}, k)$ in (1.2) implies that the $\mathbb{F}$-stopping rule τ_k of (3.2) is optimal for (2.7).

Other Cases. In the remainder we shall assume either $\lambda_1 > \lambda_0$, $0 < \phi_d < k$ (**Case II**) or $\lambda_1 < \lambda_0$ (**Case III**). Unlike Case I above, the process $\Phi^{(\alpha)}$ may now return to the interval $[0, k]$ with positive probability after every exit; see Figure 1(a) with k' instead of k, and

Figure 1(c). However, we shall show that the optimal stopping rule for (2.7) is still of the form $\tau_\phi \triangleq \inf\{t \geq 0 : \Phi_t^{(\alpha)} \geq \phi\}$ for some suitable $\phi > k$. In terms of the auxiliary discounted optimal stopping problem

$$U(\phi; \Phi^{(\alpha)}, k) \ \triangleq \ \inf_{\tau \in \mathcal{S}} \ \mathbb{E}_0^\phi \int_0^\tau e^{-\lambda t}(\Phi_t^{(\alpha)} - k)\, dt, \quad \phi \in \mathbb{R}_+, \tag{3.4}$$

where $\mathbb{E}_0^\phi$ is the expectation under $\mathbb{P}_0$ given that $\Phi_0^{(\alpha)} = \phi \in \mathbb{R}_+$, the minimum Bayes risk in (2.7) can be written as

$$V(\pi; \Phi^{(\alpha)}, k) = \gamma(\pi) + \beta(\pi) \cdot U\left(\frac{\pi}{1-\pi}; \Phi^{(\alpha)}, k\right), \quad \pi \in [0, 1). \tag{3.5}$$

Since one can always stop immediately, and the process $\Phi^{(\alpha)}$ is nonnegative, we have $-k/\lambda \leq U(\phi; \Phi^{(\alpha)}, k) \leq 0$, $\phi \in \mathbb{R}_+$, i.e., the value function $U(\,\cdot\,; \Phi^{(\alpha)}, k)$ in (3.4) is bounded.

Lemma 3.1 (Verification Lemma). *Let $g : \mathbb{R}_+ \mapsto (-\infty, 0]$ be a bounded, continuous and piecewise continuously differentiable function such that*

$$[\lambda + ay]g'(y) - bg(y) + \lambda_0 g(ry) \geq -y + k, \qquad y \in \mathbb{R}_+ \tag{3.6}$$

whenever $g'(y)$ exists. Then $U(y; \Phi^{(\alpha)}, k) \geq g(y)$ for every $y \in \mathbb{R}_+$.

In addition, if $g \in \mathcal{C}(\mathbb{R}_+) \cap \mathcal{C}^1(\mathbb{R}_+ \backslash \{\phi_d, \phi\})$ for some real number $\phi > k$ and

$$[\lambda + ay]g'(y) - bg(y) + \lambda_0 g(ry) = -y + k, \qquad y \in (0, \phi_d) \cup (\phi_d, \phi), \tag{3.7}$$

$$g(y) = 0, \qquad y \in [\phi, \infty), \tag{3.8}$$

then we have $U(y; \Phi^{(\alpha)}, k) = g(y)$ for every $y \in \mathbb{R}_+$. The $\mathbb{F}$-stopping time

$$\tau_\phi \triangleq \inf\{t \geq 0 : \Phi_t^{(\alpha)} \geq \phi\} \tag{3.9}$$

is optimal for (3.4) and (2.7) and has finite $\mathbb{P}_0$-expectation.

Proof. By the chain-rule, we have

$$\begin{aligned} e^{-\lambda\tau} g(\Phi_\tau^{(\alpha)}) = g(\Phi_0^{(\alpha)}) &+ \int_0^\tau e^{-\lambda s}(\mathcal{A}^{(\alpha)} - \lambda) g(\Phi_s^{(\alpha)}) ds \\ &+ \int_0^\tau e^{-\lambda s}[g(r\Phi_{s-}^{(\alpha)}) - g(\Phi_{s-}^{(\alpha)})](dX_s - \lambda_0 ds), \qquad \tau \in \mathcal{S}, \end{aligned} \tag{3.10}$$

where $\mathcal{A}^{(\alpha)}$ is the infinitesimal generator under $\mathbb{P}_0$ of $\Phi^{(\alpha)}$ in (2.17). Since $g(\cdot)$ is bounded, the function $s \mapsto e^{-\lambda s}[g(r\Phi_{s-}^{(\alpha)}) - g(\Phi_{s-}^{(\alpha)})]$ is absolutely integrable on $\mathbb{R}_+$ with respect to the $(\mathbb{P}_0, \mathbb{F})$-compensator $s \mapsto \lambda_0 s$ of the process X. Therefore, the $\mathbb{P}_0$-expectation of the integral with respect to $X_s - \lambda_0 s$ vanishes. Since all other terms (3.10) are $\mathbb{P}_0$-integrable, so is the Lebesgue integral; especially, it is finite $\mathbb{P}_0$-almost surely. Furthermore,

$$\begin{aligned}(\mathcal{A}^{(\alpha)} - \lambda)g(y) &= [\lambda + (\lambda + \alpha - \lambda_1 + \lambda_0)y]g'(y) - \lambda_0 g(y) + \lambda_0 g((\lambda_1/\lambda_0)y) - \lambda g(y) \\ &= [\lambda + ay]g'(y) - bg(y) + \lambda_0 g(ry),\end{aligned}$$

for every $y \in \mathbb{R}_+$, see (2.17) and (3.1). After rearranging the terms in (3.10), and taking $\mathbb{P}_0$-expectations, we obtain

$$\begin{aligned}(3.11)\quad g(y) = \mathbb{E}_0[g(\Phi_0^{(\alpha)})] &= \mathbb{E}_0^y\left[e^{-\lambda\tau} g(\Phi_\tau^{(\alpha)})\right] - \mathbb{E}_0 \int_0^\tau e^{-\lambda s}(\mathcal{A}^{(\alpha)} - \lambda)g(\Phi_s^{(\alpha)})ds \\ &\leq \mathbb{E}_0 \int_0^\tau e^{-\lambda s}(\Phi_s^{(\alpha)} - k)ds, \quad \tau \in \mathcal{S},\ y \in \mathbb{R}_+,\end{aligned}$$

since $g(\cdot)$ is nonpositive and (3.6) holds. Namely, $g(y) \leq U(y; \Phi^{(\alpha)}, k)$ for every $y \in \mathbb{R}_+$. Suppose now that $g(\cdot)$ satisfies (3.7) and (3.8) for some $\phi > k$. Then (3.11) holds with an equality for the stopping time τ_ϕ of (3.9). Therefore, $g(y) = U(y; \Phi^{(\alpha)}, k)$, $y \in \mathbb{R}_+$, and the $\mathbb{F}$-stopping time τ_ϕ is optimal for (3.4). If $\Phi_0^{(\alpha)} \geq \phi$, then $\tau_\phi \equiv 0$ has finite expectation obviously. To show the same when $\Phi_0^{(\alpha)} < \phi$, note first that from (2.11) we have

$$\Phi_{t\wedge\tau_\phi}^{(\alpha)} = \Phi_0^{(\alpha)} + \int_0^{t\wedge\tau_\phi} [\lambda + (\lambda + \alpha)\Phi_s^{(\alpha)}]ds + \int_0^{t\wedge\tau_\phi} [(\lambda_1/\lambda_0) - 1]\Phi_{s-}^{(\alpha)}(dX_s - \lambda_0 ds), \quad t \geq 0,$$

where τ_ϕ is the stopping time in (3.9). The $\mathbb{P}_0$-expectation of the second integral vanishes, since $\mathbb{E}_0 \int_0^{t\wedge\tau_\phi} |[(\lambda_1/\lambda_0) - 1]\Phi_{s-}^{(\alpha)}|\lambda_0 ds \leq (\lambda_0 + \lambda_1)\phi t < \infty$. Because $\Phi^{(\alpha)}$ is nonnegative, the $\mathbb{P}_0$-expectation gives

$$[(\lambda_1/\lambda_0) + 1]\phi \geq \mathbb{E}_0\big(\Phi_{t\wedge\tau_\phi}^{(\alpha)}\big) = \mathbb{E}_0\big(\Phi_0^{(\alpha)}\big) + \mathbb{E}_0 \int_0^{t\wedge\tau_\phi} [\lambda + (\lambda + \alpha)\Phi_s^{(\alpha)}]ds \geq \lambda\, \mathbb{E}_0(t \wedge \tau_\phi), \quad t \geq 0.$$

The monotone convergence theorem implies that $\mathbb{E}_0(\tau_\phi) \leq [(\lambda_1/\lambda_0) + 1]\phi/\lambda < \infty$. □

Proposition 3.2 (**Case II and III**). *There exist a unique real number $\phi^* > k$ and a unique function $g : \mathbb{R}_+ \mapsto [-k/\lambda, 0]$ in $\mathcal{C}(\mathbb{R}_+) \cap \mathcal{C}^1(\mathbb{R}_+ \backslash \{\phi_d, \phi^*\})$, which satisfy (3.6)–(3.8) with ϕ^* instead of ϕ. The minimum Bayes risk in (2.7) is*

$$V(\pi; \Phi^{(\alpha)}, k) = \gamma(\pi) + \beta(\pi) \cdot g\left(\frac{\pi}{1-\pi}\right), \qquad \pi \in [0, 1),$$

and $\tau_{\phi^*} = \{t \geq 0 : \Phi_t^{(\alpha)} \geq \phi^*\}$ is an optimal Bayes stopping rule.

The proof of the existence and uniqueness of ϕ^* and $g(\cdot)$ is similar to that of Proposition 3.2 in Bayraktar and Dayanik (2003); the rest follows from Lemma 3.1 above. The proof of existence is by direct construction; it is summarized in two propositions below, which also yield efficient numerical methods to calculate the minimum Bayes risk and the optimal Bayes rule. Note that (3.7) is a differential-delay equation of *advanced type* in Case II ($r = \lambda_1/\lambda_0 > 1$), and it is a differential-delay equation of *retarded type* in Case III ($r = \lambda_1/\lambda_0 < 1$); see, e.g., Bellman and Cooke (1963, p. 48).

Case II: $\lambda_1 > \lambda_0$ and $0 < \phi_d < k$. For every real number $\phi > \phi_d$, denote by $h_\phi : [\phi_d, \infty) \mapsto \mathbb{R}$ the unique solution in $\mathcal{C}([\phi_d, \infty)) \cap \mathcal{C}^1([\phi_d, \phi) \cup (\phi, \infty))$ of

$$h'_\phi(y) = -\lambda_0 l(y) h_\phi(ry) - \operatorname{sgn}(\lambda + ay)|\lambda + ay|^{-b/a-1}\,(y - k)\,, \qquad y \in [\phi_d, \phi), \tag{3.12}$$

$$h_\phi(y) = 0, \qquad y \in [\phi, +\infty). \tag{3.13}$$

Here the quantity

$$l(y) \triangleq \operatorname{sgn}(\lambda + ay)|\lambda + ay|^{-b/a-1}|\lambda + ary|^{b/a}$$

is well-defined for every $y \in [\phi_d, \infty)$, since $a < 0$ and $r > 1$; see (3.1).

Proposition 3.3 (The characterization of ϕ^* and $g(\cdot)$ of Proposition 3.2 in Case II)**.** *The function $h_{\phi^*}(\cdot)$ is the only one among all $h_\phi(\cdot)$ with $\phi > \phi_d$, such that*

$$f_{-1}(y) \triangleq -(k/\lambda)|\lambda + ay|^{-b/a} \le h_\phi(y) \le 0, \qquad \forall\, y \in [\phi_d, \infty). \tag{3.14}$$

By defining $h_{\phi^*}(\cdot)$ on $(0, \phi^*)$ as the solution of the differential equation (3.12), its extension onto $\mathbb{R}_+$ (denoted also by $h_{\phi^*}(\cdot)$) remains between the same bounds of (3.14) on $\mathbb{R}_+$. We have $g(y) = |\lambda + ay|^{b/a} h_{\phi^*}(y)$ for every $y \in \mathbb{R}_+$, and

$$k < \phi^* < \overline{\phi} \triangleq (rk/\lambda)\left[(b-a)/(r^{-b/a} - 1) + (\lambda/b)\big(\,b - a - \lambda/k\,\big)\right].$$

The function $I(\phi) \triangleq h_\phi(\phi_d)$, $\phi \in [k, \infty)$ is continuous and strictly decreasing, and $I(\phi^*) = 0$.

By means of Proposition 3.3, one can find ϕ^* (and $h_{\phi^*}(\cdot)$ on $[\phi_d, \infty)$) by a *bisection search* in the interval $(k, \overline{\phi})$: At the beginning, set $(\underline{\phi}_0, \overline{\phi}_0) = (k, \overline{\phi})$. Then calculate the mid-points ϕ_n of the intervals $[\underline{\phi}_n, \overline{\phi}_n]$ for every $n \ge 0$; if $I(\phi_n) < 0$, then set $(\underline{\phi}_{n+1}, \overline{\phi}_{n+1}) = (\underline{\phi}_n, \phi_n)$, otherwise set $(\underline{\phi}_{n+1}, \overline{\phi}_{n+1}) = (\phi_n, \overline{\phi}_n)$. Then $\{\phi^*\} = \cap_{n\ge 0}[\underline{\phi}_n, \overline{\phi}_n]$. Although the solution $h_\phi(\cdot)$ of (3.12, 3.13) is unavailable in closed-form, it can be calculated on $[k, \phi]$ accurately by finite-difference methods. After ϕ^* and h_{ϕ^*} on $[\phi_d, \infty)$ have been found, h_{ϕ^*} can be calculated on $[0, \phi_d)$ from (3.12) by the continuation process (see, e.g., Bellman and Cooke (1963, p. 47)).

Case III: $\lambda_1 < \lambda_0$**.** For every real number β, let $\hbar_\beta : \mathbb{R}_+ \mapsto \mathbb{R}$ be the unique continuously differentiable solution of

$$\hbar'_\beta(y) = -(\lambda + ay)^{-b/a-1}\left[\lambda_0(\lambda + ary)^{b/a}\hbar_\beta(ry) + y - k\right], \quad y > 0, \tag{3.15}$$

$$\hbar_\beta(0) = \beta. \tag{3.16}$$

The differential equations in (3.12) and (3.15) are essentially the same (in the latter case, $\lambda + ay$ is positive for every $y \in \mathbb{R}_+$ since a is positive). However, the solution $h_\phi(y)$ of (3.12) is unique if it is initially described for all $y \in [\phi, r\phi)$, whereas $\hbar_\beta(0)$ uniquely determines the solution $\hbar_\beta(\cdot)$ of (3.15).

Proposition 3.4 (The characterization of ϕ^* and $g(\cdot)$ of Proposition 3.2 in Case III)**.** *For every $y \in [0, \phi^*)$, we have $g(y) = (\lambda + ay)^{b/a}\hbar_{\beta^*}(y)$, where β^* is the unique number satisfying both $\hbar_{\beta^*}(\phi^*) = \hbar'_{\beta^*}(\phi^*) = 0$ and*

$$f_{-1}(y) \triangleq -(k/\lambda)[\lambda + ay]^{-b/a} \le \hbar_{\beta^*}(y) \le 0, \qquad \forall\, y \in [0, \phi^*]. \tag{3.17}$$

Moreover, $k < \phi^* < bk/\lambda$ and $-k\lambda^{-b/a-1} < \beta^* < 0$. The function defined by $J(\beta) \triangleq \max_{y\in[0,bk/\lambda]} \hbar_\beta(y)$, $\beta \in [-k\lambda^{-b/a-1}, 0]$ is continuous and strictly increasing, and $J(\beta^*) = 0$.

		λ_1/λ_0					
Criterion	k	1/4	1/3	1/2	2	3	4
Linear, $R^{(1)}$ ($\varepsilon = 0.1/\lambda$)	6.7863	11.3701	10.2144	8.5541	8.8206	15.5691	25.4968
Linear, $R^{(2)}$	7.5000	12.6422	11.3458	9.4826	**9.7966**	17.3116	26.9985
Expected Miss, $R^{(3)}$	5.0000	8.1969	7.3929	**6.2366**	6.3858	11.2236	17.5763
Exponential, $R^{(4)}$ ($\alpha = 1$)	7.5000	11.6232	10.4710	8.9305	7.9989	14.1542	22.9162

Table 1. The critical thresholds ϕ^* in the definition of the optimal alarm times $\tau^* \triangleq \inf\{t \ge 0 : \Phi_t^{(\alpha)} \ge \phi^*\}$ for the Poisson disorder problem, are calculated for the criteria in (1.1) for different λ_1/λ_0 ratios ($\lambda_0 = 3$, $\lambda = 1.5$, $c = 0.20$). In the equivalent form (1.2) of those criteria, the k-values are given by Proposition 2.1, and $\alpha = 0$ for the first three criteria. In Figures 2 and 3, the details of our numerical methods are illustrated on the examples in boldface.

One can find β^* in Proposition 3.4 by bisection search in the interval $(\underline{\beta}_0, \overline{\beta}_0) = (-k\lambda^{-b/a-1}, 0)$: For every $n \geq 0$, let β_n be the mid-point of $(\underline{\beta}_n, \overline{\beta}_n)$. If $J(\beta_n) < 0$, then let $(\underline{\beta}_{n+1}, \overline{\beta}_{n+1}) = (\beta_n, \overline{\beta}_n)$, otherwise $(\underline{\beta}_{n+1}, \overline{\beta}_{n+1}) = (\underline{\beta}_n, \beta_n)$. Then $\{\beta^*\} = \cap_{n\geq 0}[\underline{\beta}_n, \overline{\beta}_n]$.

Illustrations. Table 1 gives an idea about the magnitudes of the changes in the optimal critical thresholds as λ_1/λ_0, the ratio of the arrival rates of X after and before the disorder, changes. Note that as $|\lambda_1/\lambda_0 - 1|$ becomes larger, the thresholds become larger for all criteria; namely, the continuation regions

$$[0, \phi^*) = \{\phi : U(\phi; \Phi^{(\alpha)}, k) < 0\} = \{\phi : V(\phi/(1+\phi); \Phi^{(\alpha)}, k) < \gamma(\phi/(1+\phi))\} \tag{3.18}$$

become wider; see (3.4) and (3.5). This is intuitively clear. As the quantity $|\lambda_1/\lambda_0 - 1|$ becomes larger, it is easier to differentiate the pre- and post-disorder behavior of X. Therefore, the minimum Bayes risks $V(\pi; \Phi^{(\alpha)}, k)$ in detecting the disorder time should decrease uniformly in $\pi \in [0, 1)$ and the continuation regions in (3.18) must become larger.

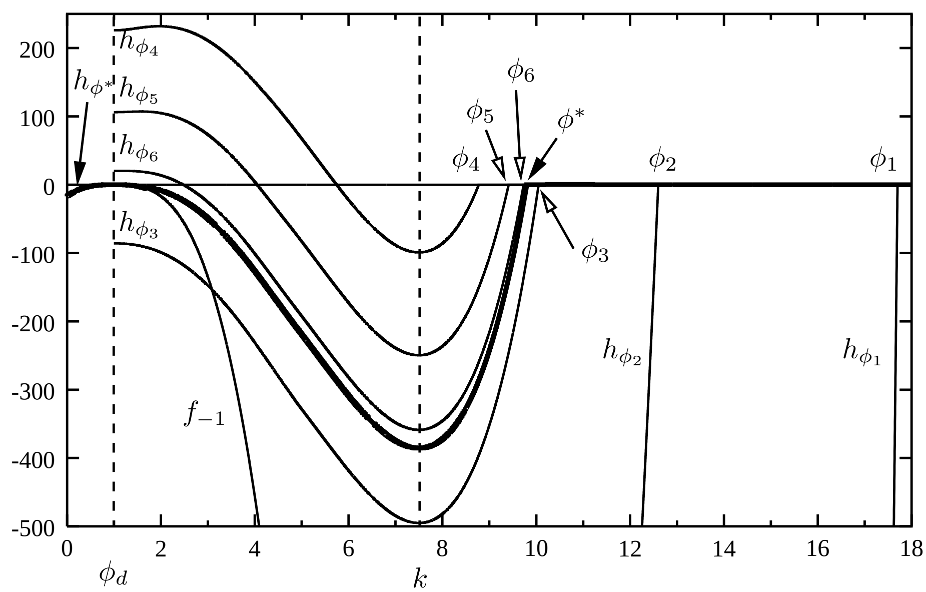

FIGURE 2. Bisection search for the critical threshold ϕ^* in Case II (see Proposition 3.3): the criterion $R^{(2)}$ in (1.1) with linear detection delay cost ($\lambda_0 = 3$, $\lambda_1 = 6$, $\lambda = 1.5$, $c = 0.20$). The search for ϕ^* starts in $(k, \overline{\phi}) = (7.500, 27.905)$ and continues along the intervals $[k, \phi_1] \supset [k, \phi_2] \supset [k, \phi_3] \supset [\phi_4, \phi_3] \supset [\phi_5, \phi_3] \supset [\phi_6, \phi_3] \supset \cdots$. The mid-points of the intervals are $\phi_1, \phi_2, \ldots$, and the search is narrowed to the lefthand (resp., righthand) half of the interval if $I(\phi_i) \triangleq h_{\phi_i}(\phi_d)$ is negative (resp., positive). The unique root of $I(\phi) = 0$ in $[\phi_d, \infty)$ is found at $\phi^* = 9.7966\cdots$ after 15 iterations.

Described after Propositions 3.3 and 3.4, the numerical methods for the calculation of the critical threshold ϕ^* in Case II and III are illustrated on two examples in Figures 2 and 3, respectively.

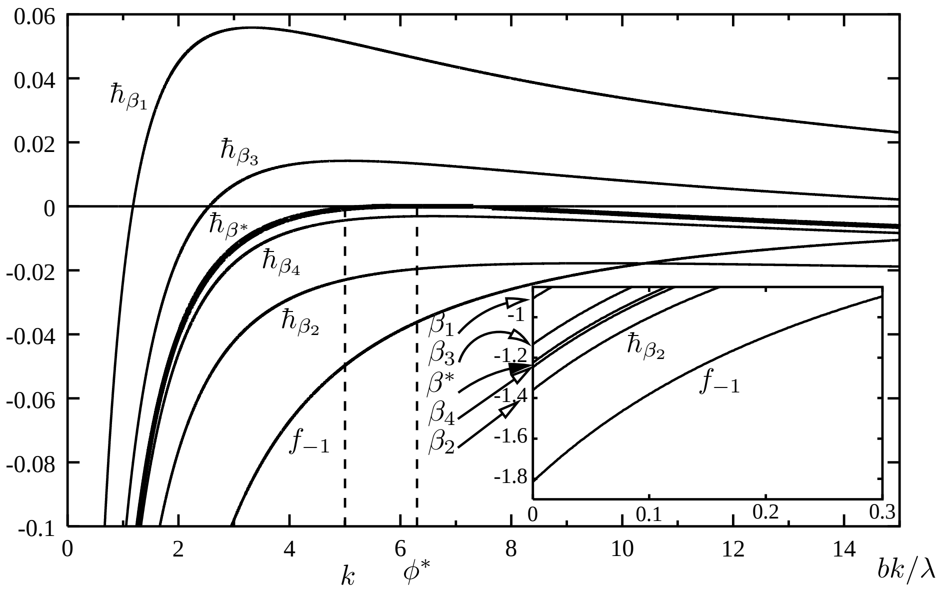

FIGURE 3. Bisection search for β^* in Case III (see Proposition 3.4): the expected total-miss criterion $R^{(3)}$ in (1.1) ($\lambda_0 = 3$, $\lambda_1 = 1.5$, $\lambda = 1.5$, $c = 0.20$). By Proposition 3.4, the critical threshold ϕ^* is contained in $(k, bk/\lambda) = (5, 15)$. Our search for β^* starts in $[k\lambda^{-b/a-1}, 0] = [-1.8144, 0]$ and continues along the intervals $[-1.8144, \beta_1] \supset [\beta_2, \beta_1] \supset [\beta_2, \beta_3] \supset [\beta_4, \beta_3] \supset \cdots$ (see the inset), where $\beta_1, \beta_2, \ldots$ are the mid-points of the intervals. At each iteration, the search for β^* continues in the lower (resp., upper) half of the interval if $J(\beta_i) \triangleq \max_{y\in[0,bk/\lambda]} \hbar_{\beta_i}(y)$ is positive (resp., negative). The unique root of $J(\beta) = 0$ in $[k\lambda^{-b/a-1}, 0]$ is found at $\beta^* = -1.2253\cdots$ after 11 iterations, and $J(\beta^*)$ is attained at $\phi^* = 6.2366\cdots$.

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